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About New Class of Univalent Starlike Functions

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ABSTRACT

In this paper we investigate and define a general class of univalent Starlike functions with respect to a convex domain contained in the right half plane. Also we establish some inclusion relationships associated with the Ruscheweyh Linear operator. Some interesting integral – preserving properties are also considered.

حول فصل جديد من الدوال النجمية وحيدة القيمة

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الكلمات المفتاحية:
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ضرب هادامارد
التبعية التفاضلية

الملخص

في هذا البحث، سو ف نتحرى ونعرف فصلاً عاماً من الدوال النجمية وحيدة القيمة بالنسبة إلى المجال المحدب في نصف المستوى الأيمن . كذلك قمنا بتأسيس بعض علاقات التضمين المرتبطة بمؤثر رشوايه الخطي. بعض الخصائص التكاملية المتعة للمؤثر المذكور قد درست

1. Introduction:

Let $h(z)$ be an analytic function with positive real part in the unit disk $U, h(0)=1$ and $h'(0)>0$, and map U onto a region starlike with respect to 1 and symmetric with respect to the real axis. Ma and Minda [1] introduced the classes $S^*(h)$ and $C(h)$ by

$$S^*(h)=\left\{f\in A: \frac{zf'(z)}{f(z)}\prec h(z)\right\}$$

$$C(h)=\left\{f\in A: 1+\frac{zf''(z)}{f'(z)}\prec h(z)\right\}$$

The classes $S^*(h)$ and $C(h)$ include the subclasses of starlike and convex functions as special cases when

$$h(z)=\frac{1+Az}{1+Bz}\quad (-1\le B<A\le 1),$$

the classes $S^*(h)$ and $C(h)$ reduce to the class $S^*[A,B]$ *of* Janowski starlike functions and the class $C[A,B]$ of Janowski convex functions respectively [2]. Thus

$$S^*[A,B]=S^*\left(\frac{1+Az}{1+Bz}\right) \text{ and } C[A,B]=C\left(\frac{1+Az}{1+Bz}\right).$$

Also

$$S^*=S^*[1,-1]=S^*\left(\frac{1+z}{1-z}\right) \text{ and } C=C[1,-1]=C\left(\frac{1+z}{1-z}\right),$$

are the familiar classes of starlike and convex functions respectively.

In this paper we investigate and consider these classes of univalent

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Starlike functions with the Ruscheweyh Linear operator.

Let $H(U)$ be the set of functions which are regular in the unit disc $U, A = \{f \in H(U): f(0) = f'(0) = 1 = 0\}$,
since $f \in A$ then may given as a series of the form

$$f(z) = z + \sum_{m=2}^{\infty} a_m z^m, \qquad (1.1)$$

Let $f_j(z)(j = 1,2)$ in A be given by'

$$f_j(z) = z + \sum_{m}^{\infty} a_{m,j} z^m,$$

Then the Hadamard product (or convolution product) $f_1 * f_2$ of $f_1(z)$ and $f_2(z)$ is defined by

$$(f_1 * f_2)(z) = z + \sum_{m=2}^{\infty} a_{m,1} a_{m,2} z^m,$$

Using the Ruscheweyh [3] Derivative D^n defined by $D^n: A \to A$;

$$D^n f(z) = \frac{z}{(1-z)^{n+1}} * f(z)$$

$$= z + \sum_{m=2}^{\infty} \frac{(n+m-1)!}{(m-1)!} a_m z^m, \ (n > -1), \qquad (1.2)$$

From (1.2), we have;

$$z\big(D^n f(z)\big)' = (n+1) D^{n+1} f(z) - n D^n f(z), \qquad (1.3)$$

Definition 1.1 [4] (Subordination Principle). For two analytic functions $f(z)$ and $F(z)$ in U we say that $f(z)$ is subordinate to $F(z)$ written symbolically as follows:

$$f \prec F \text{ in } U \text{ or } f(z) \prec \mathrm{F(z)} \quad (z \in U),$$

if there exists a Schwarz function $w(z) \in \Omega$ which (by definition) is analytic in U with, $w(0) = 0$ and
$|w(z)| < 1 \ (z \in U)$.

Remark 1.1 ([4], [5]).
If $f(z) \in S$, Ruscheweyh defined the symbol $D^n f(z)$ by;

$$D^n f(z) = \frac{z(z^{n-1} f(z))^{(n)}}{n!}, \qquad n \in N_0 = \{0,1,2, \dots\},$$

We note that:
$D^0 f(z) = f(z),\ and \quad D^1 f(z) = z f'(z), \dots$

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2. Preliminary results

We recall here the definition of the well-known class of Starlike functions:

$$S^* = \left\{ f \in A: Re\left(\frac{zf'(z)}{f(z)}\right) > 0, z \in U \right\}.$$

Let C denote the class of all convex functions in A. An analytic description of the class C is given by

$$C = \left\{ f \in A: Re\left(\frac{1 + zf''(z)}{f'(z)}\right) > 0, z \in U \right\}.$$

The classes S^* and C were first introduced by Robertson [6].

Remark 2.1

By using the subordination relation, we may define the class S^* thus if $f(z) = z + a_2 z^2 + \cdots, z \in U$, then $f \in S^*$ if and only if

$$\frac{zf'(z)}{f(z)} \prec \frac{1+z}{1-z}, \qquad z \in U.$$

Let consider the Libera-Pascu integral operator $L_a: A \to A$ defined as:

$$f(z) = L_a f(z) = \frac{1+a}{z^a} \int_0^z f(t) t^{a-1} dt, \qquad a \in \mathbb{C}, Re\ a \geq 0,$$

$$L_a f(z) = z + \sum_{n=2}^{\infty} \left(\frac{1+a}{n+a}\right) a_n z^n. \qquad (2.2)$$

In the case $a = 1$ this operator was introduced by R.J. Libera and it was studied by many authors in different general cases. In this general form $(a \in \mathbb{C}, Re\ a \geq 0)$ was used first time by N. N. Pascu in [7].

The next theorem is result of the so called "admissible functions method" introduced by P.T. Mocanu and S.S. Miller (see [8],[9],[4]).

Lemma 2.1 [10]

Let h convex in U and $R[\beta h(z) + \gamma] > 0, z \in U$. If $p \in h(U)$ with $p(0) = h(0)$ and p satisfied the Briot-Bouquet differential subordination;

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec h(z), \qquad then \qquad p(z) \prec h(z).$$

3. Main results

Definition 3.1

Let $h(z) \in H(U)$, with $h(0) = 1$ and $h(U) = D$, where D is a convex domain contained in the right half plane. We say that a function $f(z) \in A$ is in the class $S_n^*(h)$ if

$$\frac{z\big(D^n f(z)\big)'}{D^n f(z)} \prec h(z), \qquad z \in U, \qquad (3.1)$$

Remark 3.1

Geometric interpretation: $f(z) \in S_n^*(h)$ if and only if $\frac{z(D^n f(z))'}{D^n f(z)}$ take all valued in the convex domain D contained in the right half-plane.

Remark 3.2

For $n = 0$ the class Starlike functions $R\left(\frac{zf'(z)}{f(z)}\right) > 0$. For $n = 1$ the class of convex functions $R\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0$. In terms of subordination;

$$\frac{zf'(z)}{f(z)} \prec 1 + \frac{1+z}{1-z} \quad and \quad 1 + \frac{zf''(z)}{f'(z)} \prec \frac{1+z}{1-z},$$

Ali [11] and Ma. And Minda [12].

Remark 3.3

For $h_1(z) \prec h_2(z)$ we have $S_n^*(h_1) \subset S_n^*(h_2)$. From the above we obtain;

$$S_n^*(h) \subset S_n^*\left(\frac{1+z}{1-z}\right),$$

Theorem 3.1

If $f \in A$ and $n > -1$, then

$$S_n^*(h) \subset S_{n+1}^*(h)$$

Proof:

Let $f(z) \in S_n^*(h)$,

and

$$p(z) = \frac{z(D^n f(z))'}{D^n f(z)},$$

$$\frac{z\left(D^n f(z)\right)'}{D^n f(z)} = \frac{(n+1)D^{n+1} f(z) - nD^n f(z)}{D^n f(z)}$$

$$= \frac{(n+1)D^{n+1} f(z)}{D^n f(z)} - n$$

$$= p(z),$$

$$\frac{D^{n+1} f(z)}{D^n f(z)} = \frac{1}{(n+1)}(p(z)+n), \qquad (3.2)$$

Differentiating (3.2), Logarithmically with respect to z, we obtain

$$\frac{z\left(D^{n+1} f(z)\right)'}{D^{n+1} f(z)} = p(z) + \frac{zp'(z)}{p(z)+n},$$

$$p(z) + \frac{zp'(z)}{p(z)+n} \prec h(z),$$

$$\frac{z\left(D^{n+1} f(z)\right)'}{D^{n+1} f(z)} \prec h(z)$$

$$f(z) \in S_{n+1}^*(h).$$

The proof of the Theorem 3.1 is completed.

Corollary 3.1

For $n = 0$ and $h(z) = \frac{1+z}{1-z}$,we have

$$S^* = S_0^*(h) \subset S_1^*(h),$$

Theorem 3.2

Let $n > -1$, if $f(z) \in S_n^*(h)$, then $L_a f(z) \in S_n^*(h)$.

Proof:

From (2.2), we have

$$z\left(L_a f(z)\right)' = (1+a)f(z) - aL_a f(z), \qquad (3.3)$$

Setting,

$$\frac{z\left(L_a f(z)\right)'}{L_a f(z)} = p(z),$$

from (3.3), we have

$$p(z) = \frac{(1+a)f(z) - aL_a f(z)}{L_a f(z)},$$

$$\frac{p(z)+a}{1+a} = \frac{f(z)}{L_a f(z)}, \qquad (3.4)$$

By logarithmically differentiating both sides of (3.4), and multiplying by z, we have;

$$\frac{zp'(z)}{p(z)+a} = \frac{zf'(z)}{f(z)} - p(z),$$

$$p(z) + \frac{zp'(z)}{p(z)+a} = \frac{zf'(z)}{f(z)} \prec h(z).$$

By lemma (2.1), $p(z) \prec h(z)$,

$$\frac{z\left(L_a f(z)\right)'}{L_a f(z)} \prec h(z),$$

So, $L_a f(z) \in S_n^*(h)$.

The proof of the Theorem 3.2 is completed.

Conclusion

In this paper, analytic univalent functions defined on the unit disc, are studied with help of Ruscheweyh linear operator. Using this operator and the techniques of differential subordination we obtained subordination theorems. Many interesting particular cases of main them are emphasized in the form of corollaries.

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