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Actions on Polish Spaces and *p*-Continuous Functions

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ABSTRACT

The action of A on X is a map, $F: A \times X \to X$, called an f-action, which satisfies some conditions. In this article, we introduce a new type of continuity called p-continuity in the context of actions on Polish spaces. The notion of p-continuity is developed using the intrinsic properties of Polish spaces, namely, the compatible metrics and the Borel structures. We provide some results concerning the measurability, expansiveness, and extensions of p-continuous f-actions in the realm of Polish spaces. For example, It is proved, among other things, that p-continuous f-actions are Borel functions under some restrictions. The findings of this paper generalize and extend some results in the literature. The approach we present in this paper depends not only on the topological properties of the spaces but also uses results from descriptive set theory.

التأثيرعلى الفضاءات الطوبولوجية البولندية والدوال ذات الاستمر اية-p

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الكلمات المفتاحية:
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الملخص

أثر الفضاء الطوبولوجي A على فضاء آخر X هو دالة $F: A \times X \to X$ تسمى دالة الأثر-f، والتي تحقق بعض الشروط المعينة. في هذه الورقة البحثية، نقدم نوعًا جديدًا من الاستمرارية يسمى مفهوم الاستمرارية-p. نقدم بعض النتائج المتعلقة بإمكانية القياس، والتوسع، والتمديد لدالة الأثر-f ذات الاستمرارية-p، في الفضاء الطوبولوجي البولندي. على سبيل المثال، تم إثبات، من بين أشياء أخرى، أن دالة الأثر-f ذات الاستمرارية-p تحت بعض القيود هي دالة Borel. نتائج هذه الورقة هي تعميم وتوسيع لبعض النتائج في الدراسات السابقة. النهج الذي نقدمه في هذه الورقة لا يعتمد فقط على الخصائص الطوبولوجية للفضاءات، ولكنه يستخدم أيضًا نتائج من نظرية المجموعات الوصفية.

Introduction

A (left) action of a space A on a space X is a map $F: A \times X \to X$ associated with a map $f: A \to X$ that satisfies

1. $F(a, x_0) = f(a)$, for all $a \in A$,
2. $F(a_0, x) = x$, for all $x \in X$.

where a_0 and x_0 are fixed points in A and X, respectively. In this case, the map F is called an f-action [1]. This paper focuses on the f-actions associated with the actions on Polish spaces. A Polish space is a separable completely metrizable topological space. These spaces are the natural setting for descriptive set theory and its applications (see Preliminaries section).

The notion of actions of space on another is frequently of interest for its relation with homotopy classes of cyclic maps. Special cases of this notion include group actions and Gottlieb groups of a space; see for instance, [1], [2] and [3]. Recently, these notions have been used to introduce and study new generalized spaces such as in [4], [5], and [6]. Continuity of functions defined in a product domain, such as f-actions, is an important concept that has been widely studied because of its importance in most of applications in topology and other branches inofathematics ([7] and [8]).

Several different types of continuity notions have been introduced for various types of functions. For functions defined in a product domain, there are the well-known notions of joint continuity and separate continuity; see e.g. [7] and [8].

Various studies examine the concept of f-actions ([1]; [5] and [6]). However, it seems that most of the literature concerning the f-actions takes the homotopy point of view. Considering f-actions between Polish space, it is natural to question whether elements from

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descriptive set theory such as measurability and set-valued functions can be introduced to this theory. The aim of this study is to examine notions of measurability, expansiveness, and extensions associated with the homotopy actions on Polish spaces. In this paper, we define and study a new type of continuity for f-actions. Using the associated map f, we investigate, among other problems, the Borel measurability of the p-continuous f-actions. The concept of p-continuity defined in this paper is different from that of joint and separate continuity [7]. Since f-actions are functions defined in a product domain, the methods described in this paper have potential applications in other areas that have similar constructions.

In Section 3, the notion of p-continuity of f-actions is introduced, and its Borel measurability is studied. The results obtained in this study generalize and extend some well-known results. In particular, a version of Michael's well-known selection theorem [9] is proved for p-continuous f-actions. In addition, the expansiveness of f-actions is defined, and a result concerning expansive p-continuous f-actions is presented (Section 4). This extends the notion of expansive group actions, which was given in [10] in the setting of uniform spaces.

Preliminaries

A topological space, X is completely metrizable if it admits a compatible metric, d, such that (X,d) is complete. A separable completely metrizable space is called a Polish space. Polish spaces form a large class of topological spaces, having various desirable properties and applications in several areas of mathematics, see e.g. [11]. In this paper, every Polish space (X,d) is equipped with the Borel σ-algebra $\mathfrak{B}_X$ and a measure $\mu: X \to [0,\infty)$. Recall that if $(X,\mathfrak{B}_X)$ and $(Y,\mathfrak{B}_Y)$ are Borel spaces, a function $f: X \to Y$ is a Borel function if $f^{-1}(A) \in \mathfrak{B}_X$ for each Borel subset $A \in \mathfrak{B}_Y$. Continuous functions are Borel functions, but there are Borel functions that are not continuous [12].

Proposition 1 [12] *Let $(X,\mathfrak{B})$ be a Borel space and (Y,d) be a non-empty Polish space. Then:*

1. *If $Z \subseteq X$ and $f: Z \to Y$ is a Borel function, then there is a Borel function $\hat{f}: X \to Y$ extends f.*
   he graph of $f: X \to Y$, $graph(f) \subseteq X \times Y$, is Borel measurable with respect to $\mathfrak{B}_X \times \mathfrak{B}_Y$.
   et $f_n: X \to Y$ be a sequence of Borel functions. If $lim_{n\to\infty} f_n(x) = f(x)$ for each x, then f is a Borel function.

Definition 1 [13] set-valued function $\mathfrak{F}: X \to 2^Y$ is function that assigns to each $x \in X$ a subset $\mathfrak{F}(x)$ of Y, where 2^Y is the family of all the non-empty closed subsets of Y. A selection of a set-valued function $\mathfrak{F}$ is a single-valued function, $\phi: X \to Y$ that satisfies $\phi(x) \in \mathfrak{F}(x)$, for each $x \in X$.

Let $(X,\mathfrak{B})$ be a Borel space, and Y a topological space. Then, $\mathfrak{F}: X \to 2^Y$ is Borel measurable if $\mathfrak{F}^{-1}(K) \in \mathfrak{B}$ for each open subset K of Y.

P position 2 [13] *Let $(Y,\mathfrak{B})$ be a Borel space and (X,d) a Polish space. Let $\mathfrak{F}: Y \to 2^X$ be a set-valued function. Then:*

1. *If $\mathfrak{F}$ is Borel measurable and $\mathfrak{F}(y)$ is closed, then there exists a Borel selection of $\mathfrak{F}$.*
2. *$\mathfrak{F}$ is Borel measurable iff $\Gamma = \{(y,x): x \in \mathfrak{F}(y)\}$ is a Borel subset of $Y \times X$.*

The p-continuity of f-actions

Definition 2 Let (X,d) be a Polish space, and let $(A,\mathfrak{B})$ be a Boel space. An f-action $F: A \times X \to X$ is called a p-continuous function at a point (a,c) in $A \times X$ if for any positive real number ε, there exists a positive real number δ such that for every $x \in X$ and for every $a \in A$,

$$(x,c) < \delta \Rightarrow d(F(a,x),F(a,c)) < \varepsilon$$

The function F is said to be p-continuous if it is p-continuous at each point in $A \times X$.

Every continuous f-action is p-continuous, but the converse need not be true. Take, for example, the f-action $F: [\frac{1}{2},1] \times \mathbb{R} \to \mathbb{R}$ defined by

$$F(a,x) = \begin{cases} 0, & \text{if } x \leq 0; \\ ax, & \text{if } x > 0. \end{cases}$$

Here, F is p-continuous, but discontinuous at 0.

Lemma 1 *Let (X,d) be a Polish space and let $(A,\mathfrak{B})$ be a Borel space that acts on X. Let $F: A \times X \to X$ be the f-action of a Borel function $f: A \to X$. Then, the set-valued function $\gamma: A \to 2^X$, defined by*

$$\gamma(a) = \{x \in X: d(x,f(a)) \leq \varepsilon\}$$

$\varepsilon > 0$ is Borel measurable for each $a \in A$.

Proof. Consider the function composition $d \circ (f \times id_X): A \times X \to \mathbb{R}$. Then $d(x,f(a))$ is a Borel function because d, id_X and f are Borel functions. Thus,

$$\begin{aligned}\Gamma &= \{(a,x): x \in \gamma(a)\} \\ &= \{(a,x): d(x,f(a)) \leq \varepsilon\}\end{aligned} \tag{1}$$

is Borel measurable. Hence, according to Proposition 2(2), γ is Borel measurable.□

Lemma 2 *Let (X,d) be a Polish space and let $(A,\mathfrak{B})$ be a Borel space acting on X. Let $F: A \times X \to X$ be the f-action of a Borel function $f: A \to X$, and let μ be the measure on X. Then, the set-valued functions $\alpha: A \to 2^X$ and $\beta: A \to 2^X$ defined by*

$$\alpha(a) = \{x \in X: d(x,f(a)) \leq \mu(x) - \mu(f(a))\}$$

and

$$\beta(a) = \{x \in X: d(f(a),x) \leq \mu(f(a)) - \mu(x)\}$$

are Borel measurable for each $a \in A$.

Proof. Let $x \in X$, and let K_x be the set of points that satisfies $\{y \in X: d(x,y) \leq \varphi(x) - \varphi(y)\}$. Then, K is a closed subset of X.

Suppose U is an open subset of X, and let $V = X \setminus U$ be its complement in X. Let

$$\alpha^{-1}(V) = \{a \in A: \alpha(a) \subset V\}$$

Then,

$$\begin{aligned}\alpha^{-1}(V) &= \{a \in A: d(x,f(a)) \leq \varphi(x) - \varphi(f(a)), \quad x \in V\} \\ &= \textstyle\bigcap_{x\in V} \{a \in A: d(x,f(a)) \leq \varphi(x) - \varphi(f(a))\} \\ &= \textstyle\bigcap_{x\in V} f^{-1}(K_x) \\ &= f^{-1}(\textstyle\bigcap_{x\in V} K_x).\end{aligned} \tag{2}$$

Since $(\bigcap_{x\in V} K_x)$ is closed, the Borel measurability of f implies that $\alpha^{-1}(V)$ is Borel measurable. Now, $\alpha^{-1}(U) = A \setminus \alpha^{-1}(V)$. Thus, $\alpha^{-1}(U)$ is Borel. Therefore, the correspondence α is Borel measurable. A similar argument shows that β is also Borel measurable. □

Theorem 1 *Let (X,d) be a Polish space, and let $(A,\mathfrak{B})$ be a Borel space acting on X. Let $F: A \times X \to X$ be the f-action of $f: A \to X$. If f is a Borel function and F is p-continuous, then the restriction $F|_{A\times f(A)}$ is a Borel function.*

Proof. Consider the subsets $\alpha(a)$, $\beta(a)$, and $\gamma(a)$ as defined in Lemma 1 and Lemma 2. Let $\{x_n\} \subset \alpha(a)$, such that $\lim_{n\to\infty} x_n = c$. Then,

$$d(x_n,f(a)) \leq \varphi(x_n) - \varphi(f(a)) \tag{3}$$

implies that $d(c,f(a)) \leq \varphi(c) - \varphi(f(a))$, and hence, $c \in \alpha(a)$. Thus, $\alpha(a)$ is a closed subset of X. Similarly, $\beta(a)$ is a closed subset X for each $a \in A$. The set $\beta(a)$ is closed since d is continuous. Let $\mathfrak{F}: A \to 2^X$ be defined by

$$\mathfrak{F}(a) = \{x \in X: d(x,f(a)) \leq \varphi(x) - \varphi(f(a)) \leq 1\}$$

that is, $\mathfrak{F}(a) = \alpha(a) \cap \gamma(a)$ with $\varepsilon = 1$. Therefore, $\mathfrak{F}$ is Borel measurable, and $\mathfrak{F}(a)$ is a non-empty and closed subset of X. Based on Proposition 2(1), there exists a Borel selection, $g: A \to X$, such that $g(a) \in \mathfrak{F}(a)$. Let the subset of X,

$$\mathfrak{F}_n(a) = \{x \in X: d(f_{n-1}(a),x) \leq d(x,f(a)) \leq \tfrac{1}{n}\} \tag{4}$$

with $n = 2,3,\ldots$, be defined such that the set, $\mathfrak{F}_1(a) = \mathfrak{F}(a)$, and $f_1(a) = g$. Then, $\mathfrak{F}_n(a)$ is closed for all $a \in A$. If f_{n-1} is a Borel function, then F_n is Borel. Thus, based on Proposition 2(1), there is a Borel selection, $f_n: A \to X$, such that $f_n(a) \in \mathfrak{F}(a)$. Then, there is a Borel sequence, $\{f_n(a)\}$, such that

$$d(f_{n-1}(a),f_n(a)) \leq \varphi(f_{n-1}(a)) - \varphi(f_n(a)), \quad \forall n \tag{5}$$

with $\lim_{n\to\infty} f_n(a) = f(a)$ for each $a \in A$. Let G be defined as the closure:

$$G = Cl\left(\bigcup_{n=1}^{\infty} \{f_n(a): a \in A\} \cup \{f(a): a \in A\}\right)$$

which is a Polish subspace of X. Then, $F|_{A\times G}$ is p-continuous, since F is p-continuous. Thus, by Equation (5),

$$\lim_{n\to\infty} F(a, f_n(a)) = F(a, f(a)),\ \forall a \in A \quad (6)$$

is Borel. Combining Equation (6) with Proposition 1(3), $F|_{A\times f(A)}$ is obtained, which is a Borel function. □

Theorem 2 *Let Y be a closed subset of a Polish space (X, d), and let $(A, \mathfrak{B})$ be a Borel space acting on X. Let the f-action $F: A \times X \to X$ be p-continuous. Suppose the following is satisfied for each $a \in A$ and $y, y' \in Y$,*

1. $d(y, y') \le \varphi(y) - \varphi(y') \Rightarrow d(m, l) \le \varphi(m) - \varphi(l)$
2. $d(y, m) \le \varphi(y) - \varphi(m)$

where $m = F(a, y)$ and $l = F(a, y')$. Then, there exists a point $z \in Y$, such that

$$d(z, \xi(a)) \le \varphi(z) - \varphi(\xi(a)), \quad \forall a \in A$$

where $\xi: A \to Y$ such that $F(a, \xi(a)) = \xi(a)$ for each $a \in A$.

Proof. Let $A' \subset A$, and let $y_n(a) = F_n(a, y)$. From conditions (1) and (2), we have

$$d(y_{n-1}(a), y_n(a)) \le \varphi(y_{n-1}(a)) - \varphi(y_n(a)) \quad (7)$$

for $n = 1,2, \ldots$with $y_0 = y$. Then, Equation (7) implies

$$\varphi(y) \ge \varphi(y_1(a)) \ge \cdots \ge \varphi(y_n(a)) \quad (8)$$

which shows that $\{\varphi(y_n(a))\}$ is a convergent sequence. Hence, for each $\varepsilon > 0$, there exists $N \in \mathbb{N}$, such that

$$d(y_n(a), y_k(a)) \le \varphi(y_n(a)) - \varphi(y_k(a)) < \varepsilon \quad (9)$$

for every $k > n > N$. Therefore, $\{\varphi(y_n(a))\}$ is a Cauchy sequence in X. Since X is complete, $\{\varphi(y_n(a))\}$ is convergent.

Let

$$z = \begin{cases} \lim_{n\to\infty} y_n(a), & a \in A'; \\ y, & a \in A\backslash A' \end{cases} \quad (10)$$

Since F is p-continuous,

$$\begin{aligned} F(a, z) &= \lim_{n\to\infty} F(a, y_n(a)) \\ &= \lim_{n\to\infty} y_{n+1}(a) \quad (11) \\ &= z \end{aligned}$$

for each $a \in A'$.

Based on Theorem 1, $y_n(a)$ is a Borel function. Since φ is continuous, we have for $a \in A$,

$$\begin{aligned} d(y, z) &= \lim_{n\to\infty} d(y, y_n(a)) \\ &\le \lim_{n\to\infty} (\varphi(y) - \varphi(y_n(a))) \quad (12) \\ &= \varphi(y) - \varphi(z) \end{aligned}$$

Then, $d(y, z) \le \varphi(y) - \varphi(z)$. The definition of ξ yields,

$$d(y_n(a), \xi(a)) \le \varphi(y_n(a)) - \varphi(\xi(a))$$

for each n. Therefore, for $a \in A$, we obtain

$$\begin{aligned} d(z, \xi(a)) &= \lim_{n\to\infty} d(y_n(a), \xi(a)) \\ &\le \lim_{n\to\infty} (\varphi(y_n(a)) - \varphi(\xi(a))) \quad (13) \\ &= \varphi(z) - \varphi(\xi(a)) \end{aligned}$$

Thus, the result follows. □

Some properties of p-continuous functions

Recall that a space Y is said to be an absolute extensor of a space X ($Y \in AE(X)$), if whenever A is a closed subset of X, and $f: A \to Y$ is continuous, there exists a continuous extension $\hat{f}: X \to Y$ of f over X [14].

Theorem 3 *Let (X, d) and (A, d') be Polish spaces, and let $F: A \times X \to X$ be a p-continuous f-action of a Borel function $f: A \to X$. Let $g: (Z, \mathfrak{B}_Z) \to (Y, \mathfrak{B}_Y)$ be a function between Borel spaces such that Y and Z are metrizable. Assume that $A \times f(A)$ is a closed subset of Z. Let $h: X \to Y$ be a continuous surjection such that the following are satisfied:*

1. *For each $y \in Y$, the fiber $h^{-1}(y)$ is Polish with respect to the metric d on X.*
2. *For each $y \in Y$, the fiber $h^{-1}(y) \in AE(X)$.*

Then, there exists a continuous extension, $\hat{F}: Z \to X$ that extends $H: A \times f(A) \to X$.

Proof. According to Theorem 1, the restriction $H = F|_{A\times f(A)}$ is a Borel function. Let τ be the Polish topology on $A \times f(A)$. Then, by [12, Theorem 13.11], there exists a Polish topology $\tau_H \supseteq \tau$ with $\mathfrak{B}(\tau_H) = \mathfrak{B}(\tau)$ such that $H: (A \times f(A), \tau_H) \to X$ is continuous. Let $B(x; n)$ be a closed ball in Y with a radius n. Define the set

$$V_n = (A \times f(A)) \cup \{z \in Z: g(z) \in B(x; n)\}$$

for each $n \ge 0$. Suppose the map q_n is given as follows:

$$q_n: V_n \to X \quad (14)$$

which extends q_{n-1}, such that

1. $q_n(g^{-1}(B(x; r))) \subset h^{-1}(B(x; r))$
2. $q_n|_{g^{-1}(B(x;r))}$

are continuous for each ball $B(x; r)$ in Y with $r \le n$. For $n = 0$, let

$$q_0(a, x) = H(a, x) \quad (15)$$

for $(a, x) \in A \times f(A)$. Given q_n as in (14), let $B(x; n + 1) \subset Y$. Then

$$q_{n+1}|_{g^{-1}(B(x;n+1))}$$

is an extension of

$$q_n|_{g^{-1}(B(x;n))}: g^{-1}(B(x; n)) \to h^{-1}(B(x; n + 1))$$

where $h^{-1}(B(x; n + 1)) \in AE(Z)$. This defines

$$q_{n+1}: V_{n+1} \to X$$

Therefore, the family $\{q_n\}_{n\ge 0}$ induces $\hat{F}$ such that

$$g = h \circ \hat{F}$$

This proves the theorem. □

With the same hypotheses of Theorem 3, we obtain the following two results as immediate consequences:

Corollary 1 *If $A \times f(A)$ is a closed subset of $A \times X$, then there exists a continuous map $A \times X \to X$, which extends the Borel function $F|_{A\times f(A)}: A \times f(A) \to X$.*

Corollary 2 *If $Y \in AE(Z)$, then $X \in AE(Z)$.*

Proof. Suppose $Y \in AE(Z)$, then by the definition of extensors, there is an extension $g: Z \to Y$ of $h \circ H: A \times f(A) \to Y$. Using Theorem 3, there exists an extension of $H: A \times f(A) \to X$ which implies that $X \in AE(Z)$. □

Definition 3 The action of a space A on a space X is called expansive if for $\varepsilon > 0$, *for all x, y in X such that $x \neq y$, there exists $a \in A$ such that*

$$d(F(a, x), F(a, y)) \ge \varepsilon$$

where, $F: A \times X \to X$ is an f-action.

Theorem 4 *Let a Borel space $(A, \mathfrak{B}_A)$ act expansively on a Polish space (X, d) such that $F: A \times X \to X$ is a p-continuous f-action. Assume that K is a closed subset of X such that the image of $F|_{A\times K}$ is a dense subset of X. Then, for all $x,\ y \in K$, there exists $a \in A$ such that*

$$d(F(a, x), F(a, y)) = d(x, y)$$

Proof. From the expansiveness of F we have for all $x,\ y \in K$,

$$d(F(a, x), F(a, y)) \ge d(x, y) \quad (16)$$

Let $\varepsilon > 0$ be given. Let $\{x_n\}$ be a convergent sequence in K such that for each $i \neq j$, $d(x_i, x_j) > \varepsilon$, where $x_i,\ x_j \in \{x_n\}$. Let $\{F(a, x_n)\}$ be the corresponding sequence in X. Thus, there exists $\delta > \varepsilon$ such that

$$d(x_i, x_j) \le d(F(a, x_i), F(a, x_j)) < \delta d(x_i, x_j) \quad (17)$$

Let $(a, y), (a, z) \in A \times K$. Then, there exist $i,\ j$ such that

$$d(F(a, y), F(a, x_i)) \le \frac{\varepsilon}{2}$$

$$d(F(a, z), F(a, x_j)) \le \frac{\varepsilon}{2}$$

and

$$d(y, x_i) \le \frac{\varepsilon}{2}$$

$$d(z, x_j) \leq \frac{\varepsilon}{2}.$$

Therefore,

$$d(F(a,y), F(a,z)) \leq d(F(a,x_i), F(a,x_j)) + \varepsilon \qquad (18)$$

and

$$d(x_i, x_j) \leq d(y,z) + \varepsilon \qquad (19)$$

Combining Equation (17), Equation (18), and Equation (19), we obtain

$$\begin{aligned} d(F(a,y), F(a,z)) &\leq d(F(a,x_i), F(a,x_j)) + \varepsilon \\ &< \delta d(x_i, x_j) + \varepsilon \qquad (20) \\ &\leq \delta(d(y,z) + \varepsilon) + \varepsilon \end{aligned}$$

Since $\varepsilon > 0$ and $\delta > \varepsilon$ were chosen arbitrarily,

$$d(F(a,y), F(a,z)) \leq d(y,z) \qquad (21)$$

The result follows from Equation (16) and Equation (21). □

Conclusion

The concept of an action of a space on another are related to several notions in topology and homotopy theory such as fibrations and H-spaces [2],[3] and [6]. The present article deals with the initiation and study of a p-continuity notion in the context of actions on Polish spaces. Firstly, we introduced in Definition 2 the notion of p-continuity of f-actions. In Theorem 1 the author developed a relation between Borel measurability and p-continuity using set-valued funtions. Using this idea, the properties of maps extensions (Theoem 3) and action expansiveness (Theorem 4) are examined for p-continuous f-actions.

The current paper has produced a new concept of continuity that combines effectively properties of Polish spaces from descriptive set theory and properties of actions from homotopy theory. Moreover, a kind of generalization has been done to the already known structures.

References

[1]- Arkowitz, M. and Lupton, G. (2005). Homotopy actions, cyclic maps and their duals, Homology Homotopy Appl. **7**(1), 169-184.

[2]- Gottlieb, D. H. (1969). Evaluation subgroups of homotopy groups, Amer. J. Math. **91**(3), 729-756. DOI: https://doi.org/10.2307/2373349

[3]- Varadarajan, K. (1969). Generalised gottlieb groups, J. Indian Math. Soc. **33**, 141-164.

[4]- Choi, H. W., Kim, J. R. and Oda, N. (2017). The generalized coGottlieb groups, related actions and exact sequences, J. Korean Math. Soc. **54**(5), 1623-1639. DOI: 10.4134/JKMS.J160602

[5]- Golasinski, M. and De Melo, T. (2019). Generalized Gottlieb and Whitehead center groups of space forms, Homology Homotopy Appl. **21**(1), 323-340. DOI: https://dx.doi.org/10.4310/HHA.2019.v21.n1.a15

[6]- Iwase, N., Mimura, M., Oda, N. and Yoon, Y. S. (2012). The Milnor-Stasheff filtration on spaces and generalized cyclic maps, Canad. Math. Bull. **55**(3), 523-536. DOI: https://doi.org/10.4153/CMB-2011-130-8

[7]- Namioka, I. (1974). Separate continuity and joint continuity, Pacific J. Math. **51**(2), 515-531.

[8]- Piotrowski, Z. (1985). Separate and joint continuity, Real Anal. Exch. **11**(2), 293-322.

[9]- Michael, E. (1956). Continuous selections II, Ann. of Math. 562-580.

[10]- Gromov, M. (1999). Endomorphisms of symbolic algebraic varieties, J. Eur. Math. Soc. **1**(2), 109-197. DOI: https://doi.org/10.1007/PL00011162

[11]- Khalil, A. and Ghafur, A. (2018). A Baues fibbration category of p-spaces, J. King Saud Univ. Sci. **30**(3), 324-329. DOI: http://dx.doi.org/10.1016/j.jksus.2016.11.008

[12]- Kechris, A., Classical descriptive set theory, Vol. 156, Springer Science & Business Media. 2012.

[13]- Aliprantis, C. D. and Border, K., Infinit Dimensional Analysis: A Hitchhiker's Guide, Springer Science & Business Media. 2006. DOI: https://doi.org/10.1007/3- 540-29587-9

[14]- Repovs, D. and Semenov, P. V., Continuous selections of multivalued mappings, Vol. 455, Springer Science & Business Media. 2013.