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A Class of Seventh Order Hybrid Extended Block Adams Moulton Methods for Numerical Solutions of First Order Delay Differential Equations

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ABSTRACT

A class of seventh order Hybrid Extended Block Adams Moulton Methods (HEBAMM) is developed for the approximate solution of some first order delay differential equations (DDEs) without the introduction of interpolation formula in calculating the delay term. The delay term was evaluated by a valid expression of sequence. By matrix inversion techniques, the discrete schemes of the proposed method were obtained through its continuous derivations with the help of linear multistep collocation procedure. The convergence and stability analysis of the method were investigated. The results obtained show that the higher step number k = 4 performed better and faster than the lower step numbers k = 3 and 2 when compared with the exact solutions and other existing methods at fixed step size e .

معادلة من الرتبة السابعة الهجينة الممتدة طرق آدمز مولتون للحلول العددية للمعادلات التفاضلية لتأخير الرتبة الأولى

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الكلمات المفتاحية:
معادلات تفاضلية تأخير من الدرجة الأولى
طريقة الكتلة الهجينة
نقاط خارج الشبكة
نقاط مستقبلية ممتدة
طريقة Adams Moulton.

الملخص

تم تطوير فئة من طرق Hbrid Extended Block Extended Block Adams Moulton (HEBAMM) من الدرجة السابعة للحل التقريبي لبعض معادلات تفاضلية التأخير من الدرجة الأولى (DDEs) دون إدخال صيغة الاستيفاء في حساب مصطلح التأخير. تم تقييم مصطلح التأخير من خلال تعبير صالح للتسلسل. من خلال تقنيات انعكاس المصفوفة ، تم الحصول على المخططات المنفصلة للطريقة المقترحة من خلال اشتقاقاتها المستمرة بمساعدة إجراء التجميع الخطي متعدد الخطوات. تم التحقيق في تقارب وتحليل الاستقرار للطريقة. تظهر النتائج التي تم الحصول عليها أن رقم الخطوة الأعلى k = 4 كان يؤدي بشكل أفضل وأسرع من أرقام الخطوة السفلية k = 3 و 2 عند مقارنته بالحلول الدقيقة والطرق الأخرى الموجودة بحجم خطوة ثابت

Introduction:

Numerically, Scholars [1, 2, 3, 4, 5, 6] have solved some delay differential equations using interpolation techniques and encountered some challenges. One of the difficulties encountered by these scholars in the use of interpolation techniques to evaluate the delay term of DDEs was studied by [7] that the order of the interpolating polynomials should be at least equal to the computational method applied in solving DDEs to which is very difficult to arrive at; if not, the accuracy of the method will not be preserved.

In order to overcome the difficulty posed by using interpolation formula in computing the delay term, we shall apply the valid expression

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formulated by [8]. This approach has been successfully applied by [9, 10, 11, 12, 13, 14, 15] for numerical approximation of first order DDEs without applying interpolation formula in calculating the delay term.
In this research work, we shall formulate and apply hybrid extended block Adams Moulton Methods in solving some first order DDEs as developed by [16]

$$y'(t) = f(t, y(t), y(t-\tau)), \quad \text{for } t > t_0, \tau > 0$$
$$y(t) = a(t), \quad \text{for } t \le t_0$$
(1)

where $a(t)$ is the initial function, τ is called the delay, $(t-\tau)$ is called the delay term and $y(t-\tau)$ is the solution of the delay term. The results obtained after the implementation of the new method shall be compared to [8, 9, 13] to prove its advantage.

Derivation of Linear Multistep Collocation Approach

The *k*-step linear multistep collocation approach with *m* collocation points was formulated in [16] as;

$$y(x) = \sum_{w=0}^{r-1} \alpha_w(x) y_{z+w} + e \sum_{w=0}^{u-1} \beta_w(x) f_{z+w}(x, y(x)) \quad (2)$$

From (2) the continuous derivations of extended and hybrid extended block Adams Moulton Methods can be respectively expressed as

$$y(x) = \sum_{w=0}^{r-1} \alpha_w(x) y_{z+w} + e \sum_{w=0}^{u-1} \beta_w(x) f_{z+w}(x, y(x)) + e \sum_{w=0}^{u-1} \gamma_w(x) g_{z+w}(x, y(x)) \quad (3)$$

$$y(x) = \sum_{w=0}^{r-1} \alpha_w(x) y_{z+w} + e \sum_{w=0}^{u-1} \beta_w(x) f_{z+w}(x, y(x)) + e \sum_{w=0}^{u-1} \gamma_w(x) g_{z+w}(x, y(x)) + e \sum_{w=0}^{u-1} \delta_w(x) l_{z+w}(x, y(x)) \quad (4)$$

where $\alpha_w(x), \beta_w(x), \gamma_w(x)$ and $\delta_w(x)$ are continuous coefficients of the technique defined as

$$\alpha_w(x) = \sum_{g=0}^{r+u-1} \alpha_{w,g+1} x^g \text{ for } w = \{0,1,\ldots,r-1\} \quad (5)$$

$$e\beta_w(x) = \sum_{g=0}^{r+u-1} e\beta_{w,g+1} x^g \text{ for } w = \{0,1,\ldots,u-1\} \quad (6)$$

$$e\gamma_w(x) = \sum_{g=0}^{r+u-1} e\gamma_{w,g+1} x^g \text{ for } w = \{0,1,\ldots,u-1\} \quad (7)$$

$$e\gamma_w(x) = \sum_{g=0}^{r+u-1} e\gamma_{w,g+1} x^g \text{ for } w = \{0,1,\ldots,u-1\} \quad (8)$$

where $w = 0,1,2,\ldots,u-1$ are the u collocation points, $x_{z+w}, w = 0,1,2,\ldots,r-1$ are the r arbitrarily picked interpolation points and e is the fixed step width.
To get $\alpha_w(x), \beta_w(x), \gamma_w(x)$ and $\delta_w(x)$, [17] developed a matrix equation of the form

$$SE = \mathrm{I} \quad (9)$$

where I is the square matrix of dimension $(r+u)\times(r+u)$ while E and S are matrices defined as

$$E = \begin{bmatrix} 1 & x_z & x_z^2 & \cdots & x_z^{r+u-1} \\ 1 & x_{z+1} & x_{z+1}^2 & \cdots & x_{z+1}^{r+u-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_{z+r-1} & x_{z+r-1}^2 & \cdots & x_{z+r-1}^{r+u-1} \\ 0 & 1 & 2x_0 & \cdots & (r+u-1)x_0^{r+u-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & 2x_{z-1} & \cdots & (r+u-1)x_{z-1}^{r+u-2} \end{bmatrix} \quad (10)$$

$$S=\begin{bmatrix} \alpha_{0,1} & \alpha_{1,1} & \cdots & \alpha_{r-1,1} & e\beta_{0,1} & \cdots & e\beta_{u-1,1} \\ \alpha_{0,2} & \alpha_{1,2} & \cdots & \alpha_{r-1,2} & e\beta_{0,2} & \cdots & e\beta_{u-1,2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \alpha_{0,r+u} & \alpha_{1,r+u} & \cdots & \alpha_{r-1,r+u} & e\beta_{0,r+u} & \cdots & e\beta_{u-1,r+u} \end{bmatrix} \tag{11}$$

From the matrix equation (9), the columns of $S=E^{-1}$ give the continuous coefficients of the continuous scheme of (4).

Derivation of HEBAMM Method with Integrated Three Off-grids Points for $k=2$

Here, we integrated three off-grids extended future points at $x=x_{z+\frac{9}{4}}, x=x_{z+\frac{5}{2}}, x=x_{z+\frac{11}{4}}$ and one extended future point at $x=x_{z+3}$ as collocation points, thus the interpolation points, $r=1$ and the collocation points $u=7$ are considered, therefore, (4) becomes

$$y(x)=\alpha_1(x)y_{z+1}+e[\beta_0(x)f_z+\beta_1(x)f_{z+1}+\beta_2(x)f_{z+2}+\beta_{\frac{9}{4}}(x)f_{z+\frac{9}{4}}+\beta_{\frac{5}{2}}(x)f_{z+\frac{5}{2}}+\beta_{\frac{11}{4}}(x)f_{z+\frac{11}{4}}+\beta_3(x)f_{z+3}] \tag{12}$$

The matrix E in (9) becomes

$$E=\begin{pmatrix} 1 & x_z+e & (x_z+e)^2 & (x_z+e)^3 & (x_z+e)^4 & (x_z+e)^5 & (x_z+e)^6 & (x_z+e)^7 \\ 0 & 1 & 2x_z & 3x_z^2 & 4x_z^3 & 5x_z^4 & 6x_z^5 & 7x_z^6 \\ 0 & 1 & 2x_z+2e & 3(x_z+e)^2 & 4(x_z+e)^3 & 5(x_z+e)^4 & 6(x_z+e)^5 & 7(x_z+e)^6 \\ 0 & 1 & 2x_z+4e & 3(x_z+2e)^2 & 4(x_z+2e)^3 & 5(x_z+2e)^4 & 6(x_z+2e)^5 & 7(x_z+2e)^6 \\ 0 & 1 & 2x_z+\frac{9}{2}e & 3\left(x_z+\frac{9}{4}e\right)^2 & 4\left(x_z+\frac{9}{4}e\right)^3 & 5\left(x_z+\frac{9}{4}e\right)^4 & 6\left(x_z+\frac{9}{4}e\right)^5 & 7\left(x_z+\frac{9}{4}e\right)^6 \\ 0 & 1 & 2x_z+5e & 3\left(x_z+\frac{5}{2}e\right)^2 & 4\left(x_z+\frac{5}{2}e\right)^3 & 5\left(x_z+\frac{5}{2}e\right)^4 & 6\left(x_z+\frac{5}{2}e\right)^5 & 7\left(x_z+\frac{5}{2}e\right)^6 \\ 0 & 1 & 2x_z+\frac{11}{2}e & 3\left(x_z+\frac{11}{4}e\right)^2 & 4\left(x_z+\frac{11}{4}e\right)^3 & 5\left(x_z+\frac{11}{4}e\right)^4 & 6\left(x_z+\frac{11}{4}e\right)^5 & 7\left(x_z+\frac{11}{4}e\right)^6 \\ 0 & 1 & 2x_z+6e & 3(x_z+3e)^2 & 4(x_z+3e)^3 & 5(x_z+3e)^4 & 6(x_z+3e)^5 & 7(x_z+3e)^6 \end{pmatrix} \tag{13}$$

The inverse of the matrix $S=E^{-1}$ is examined using Maple 18 from which the continuous scheme is obtained using (4), evaluating and simplifying it at $x=x_z, x=x_{z+2}, x=x_{z+\frac{9}{4}}, x=x_{z+\frac{5}{2}}, x=x_{z+\frac{11}{4}}, x=x_{z+3}$,the following discrete schemes are obtained

$$\begin{aligned}
y_z&=y_{z+1}-\frac{67031}{249480}ef_z-\frac{6037}{3528}ef_{z+1}+\frac{47129}{2520}ef_{z+2}-\frac{130048}{2835}ef_{z+\frac{9}{4}}+\frac{14296}{315}ef_{z+\frac{5}{2}}-\frac{510464}{24255}ef_{z+\frac{11}{4}}+\frac{28817}{7560}ef_{z+3}\\
y_{z+2}&=y_{z+1}-\frac{799}{249480}ef_z+\frac{543}{1960}ef_{z+1}+\frac{3307}{840}ef_{z+2}-\frac{4096}{567}ef_{z+\frac{9}{4}}+\frac{216}{35}ef_{z+\frac{5}{2}}-\frac{20992}{8085}ef_{z+\frac{11}{4}}+\frac{3313}{7560}ef_{z+3}\\
y_{z+\frac{9}{4}}&=y_{z+1}-\frac{40825}{12773376}ef_z-\frac{250055}{903168}ef_{z+1}+\frac{520375}{129024}ef_{z+2}-\frac{63755}{9072}ef_{z+\frac{9}{4}}+\frac{12325}{2016}ef_{z+\frac{5}{2}}-\frac{200125}{77616}ef_{z+\frac{11}{4}}+\frac{168625}{387072}ef_{z+3}\\
y_{z+\frac{5}{2}}&=y_{z+1}-\frac{43}{13440}ef_z+\frac{1737}{6272}ef_{z+1}+\frac{18021}{4480}ef_{z+2}-\frac{724}{105}ef_{z+\frac{9}{4}}+\frac{219}{35}ef_{z+\frac{5}{2}}-\frac{636}{245}ef_{z+\frac{11}{4}}+\frac{1961}{4480}ef_{z+3}\\
y_{z+\frac{11}{4}}&=y_{z+1}-\frac{5831}{1824768}ef_z+\frac{567}{2048}ef_{z+1}+\frac{123823}{30720}ef_{z+2}-\frac{44933}{6480}ef_{z+\frac{9}{4}}+\frac{1029}{160}ef_{z+\frac{5}{2}}-\frac{6559}{2640}ef_{z+\frac{11}{4}}+\frac{119707}{276480}ef_{z+3}\\
y_{z+3}&=y_{z+1}-\frac{20}{6237}ef_z+\frac{611}{2205}ef_{z+1}+\frac{1264}{315}ef_{z+2}-\frac{19456}{2835}ef_{z+\frac{9}{4}}+\frac{1984}{315}ef_{z+\frac{5}{2}}-\frac{54272}{24255}ef_{z+\frac{11}{4}}+\frac{487}{945}ef_{z+3}
\end{aligned} \tag{14}$$

Derivation of HEBAMM Method with Integrated Two Off-grids Points for $k=3$

In this case, we integrated two off-grids extended future points at $x=x_{z+\frac{7}{2}}, x=x_{z+\frac{15}{4}}$ and one extended future point at $x=x_{z+4}$ as collocation points, thus the interpolation points, $r=1$ and the collocation points $u=7$ are considered, therefore, (4) becomes

$$y(x)=\alpha_2(x)y_{z+2}+e[\beta_0(x)f_z+\beta_1(x)f_{z+1}+\beta_2(x)f_{z+2}+\beta_3(x)f_{z+3}+\beta_{\frac{7}{2}}(x)f_{z+\frac{7}{2}}+\beta_{\frac{15}{4}}(x)f_{z+\frac{15}{4}}+\beta_4(x)f_{z+4}] \tag{15}$$

The matrix E in (9) becomes

$$E=\begin{pmatrix} 1 & x_z+2e & (x_z+2e)^2 & (x_z+2e)^3 & (x_z+2e)^4 & (x_z+2e)^5 & (x_z+2e)^6 & (x_z+2e)^7 \\ 0 & 1 & 2x_z & 3x_z^2 & 4x_z^3 & 5x_z^4 & 6x_z^5 & 7x_z^6 \\ 0 & 1 & 2x_z+2e & 3(x_z+e)^2 & 4(x_z+e)^3 & 5(x_z+e)^4 & 6(x_z+e)^5 & 7(x_z+e)^6 \\ 0 & 1 & 2x_z+4e & 3(x_z+2e)^2 & 4(x_z+2e)^3 & 5(x_z+2e)^4 & 6(x_z+2e)^5 & 7(x_z+2e)^6 \\ 0 & 1 & 2x_z+6e & 3(x_z+3e)^2 & 4(x_z+3e)^3 & 5(x_z+3e)^4 & 6(x_z+3e)^5 & 7(x_z+3e)^6 \\ 0 & 1 & 2x_z+7e & 3\left(x_z+\frac{7}{2}e\right)^2 & 4\left(x_z+\frac{7}{2}e\right)^3 & 5\left(x_z+\frac{7}{2}e\right)^4 & 6\left(x_z+\frac{7}{2}e\right)^5 & 7\left(x_z+\frac{7}{2}e\right)^6 \\ 0 & 1 & 2x_z+\frac{15}{2}e & 3\left(x_z+\frac{15}{4}e\right)^2 & 4\left(x_z+\frac{15}{4}e\right)^3 & 5\left(x_z+\frac{15}{4}e\right)^4 & 6\left(x_z+\frac{15}{4}e\right)^5 & 7\left(x_z+\frac{15}{4}e\right)^6 \\ 0 & 1 & 2x_z+8e & 3(x_z+4e)^2 & 4(x_z+4e)^3 & 5(x_z+4e)^4 & 6(x_z+4e)^5 & 7(x_z+4e)^6 \end{pmatrix} \tag{16}$$

The inverse of the matrix $S=E^{-1}$ is examined using Maple 18 from which the continuous scheme is obtained using (4), evaluating and simplifying it at $x=x_z, x=x_{z+1}, x=x_{z+3}, x=x_{z+\frac{7}{2}}, x=x_{z+\frac{15}{4}}, x=x_{z+4}$,the following discrete schemes are obtained

$$\begin{aligned}
y_z &= y_{z+2}-\frac{3841}{13230}ef_z-\frac{5506}{3465}ef_{z+1}+\frac{962}{2205}ef_{z+2}-\frac{2582}{945}ef_{z+3}+\frac{14464}{2205}ef_{z+\frac{7}{2}}-\frac{434176}{72765}ef_{z+\frac{15}{4}}+\frac{997}{630}ef_{z+4}\\
y_{z+1} &= y_{z+2}+\frac{127}{13230}ef_z-\frac{3461}{9240}ef_{z+1}-\frac{5659}{5880}ef_{z+2}+\frac{9307}{7560}ef_{z+3}-\frac{1864}{735}ef_{z+\frac{7}{2}}+\frac{159232}{72765}ef_{z+\frac{15}{4}}-\frac{467}{840}ef_{z+4}\\
y_{z+3} &= y_{z+2}+\frac{2}{1323}ef_z-\frac{551}{27720}ef_{z+1}+\frac{6967}{17640}ef_{z+2}+\frac{8963}{7560}ef_{z+3}-\frac{3032}{2205}ef_{z+\frac{7}{2}}+\frac{77312}{72765}ef_{z+\frac{15}{4}}-\frac{629}{2520}ef_{z+4}\\
y_{z+\frac{7}{2}} &= y_{z+2}+\frac{181}{125440}ef_z-\frac{471}{24640}ef_{z+1}+\frac{24429}{62720}ef_{z+2}+\frac{391}{280}ef_{z+3}-\frac{927}{980}ef_{z+\frac{7}{2}}+\frac{2416}{2695}ef_{z+\frac{15}{4}}-\frac{3921}{17920}ef_{z+4}\\
y_{z+\frac{15}{4}} &= y_{z+2}+\frac{6419}{4423680}ef_z-\frac{77861}{4055040}ef_{z+1}+\frac{287567}{737280}ef_{z+2}+\frac{1536983}{1105920}ef_{z+3}-\frac{9359}{11520}ef_{z+\frac{7}{2}}+\frac{1526}{1485}ef_{z+\frac{15}{4}}-\frac{335111}{1474560}ef_{z+4}\\
y_{z+4} &= y_{z+2}+\frac{19}{13230}ef_z-\frac{2}{105}ef_{z+1}+\frac{286}{735}ef_{z+2}+\frac{1322}{945}ef_{z+3}-\frac{128}{147}ef_{z+\frac{7}{2}}+\frac{8192}{6615}ef_{z+\frac{15}{4}}-\frac{29}{210}ef_{z+4}
\end{aligned} \tag{17}$$

Construction of HEBAMM Method with Integrated One Off-grid Point for $k=4$

With the same procedure, we integrated one off-grid extended future points at $x=x_{z+\frac{9}{2}}$ and one extended future point at $x=x_{z+5}$ as collocation points, thus the interpolation points, $r=1$ and the collocation points $u=7$ are considered, therefore, (4) becomes

$$y(x)=\alpha_3(x)y_{z+3}+e[\beta_0(x)f_z+\beta_1(x)f_{z+1}+\beta_2(x)f_{z+2}+\beta_3(x)f_{z+3}+\beta_4(x)f_{z+4}+\beta_{\frac{9}{2}}(x)f_{z+\frac{9}{2}}+\beta_5(x)f_{z+5}] \tag{18}$$

The matrix E in (9) becomes

$$E=\begin{pmatrix} 1 & x_z+3e & (x_z+3e)^2 & (x_z+3e)^3 & (x_z+3e)^4 & (x_z+3e)^5 & (x_z+3e)^6 & (x_z+3e)^7 \\ 0 & 1 & 2x_z & 3x_z^2 & 4x_z^3 & 5x_z^4 & 6x_z^5 & 7x_z^6 \\ 0 & 1 & 2x_z+2e & 3(x_z+e)^2 & 4(x_z+e)^3 & 5(x_z+e)^4 & 6(x_z+e)^5 & 7(x_z+e)^6 \\ 0 & 1 & 2x_z+4e & 3(x_z+2e)^2 & 4(x_z+2e)^3 & 5(x_z+2e)^4 & 6(x_z+2e)^5 & 7(x_z+2e)^6 \\ 0 & 1 & 2x_z+6e & 3(x_z+3e)^2 & 4(x_z+3e)^3 & 5(x_z+3e)^4 & 6(x_z+3e)^5 & 7(x_z+3e)^6 \\ 0 & 1 & 2x_z+8e & 3(x_z+4e)^2 & 4(x_z+4e)^3 & 5(x_z+4e)^4 & 6(x_z+4e)^5 & 7(x_z+4e)^6 \\ 0 & 1 & 2x_z+9e & 3\left(x_z+\frac{9}{2}e\right)^2 & 4\left(x_z+\frac{9}{2}e\right)^3 & 5\left(x_z+\frac{9}{2}e\right)^4 & 6\left(x_z+\frac{9}{2}e\right)^5 & 7\left(x_z+\frac{9}{2}e\right)^6 \\ 0 & 1 & 2x_z+10e & 3(x_z+5e)^2 & 4(x_z+5e)^3 & 5(x_z+5e)^4 & 6(x_z+5e)^5 & 7(x_z+5e)^6 \end{pmatrix} \tag{19}$$

The inverse of the matrix $S=E^{-1}$ is examined using Maple 18 from which the continuous scheme is derived using (4), evaluating and simplifying it at $x=x_z, x=x_{z+1}, x=x_{z+2}, x=x_{z+4}, x=x_{z+\frac{9}{2}}, x=x_{z+5}$,the following discrete schemes are obtained

$$y_z = y_{z+3} - \frac{1013}{3360} e f_z - \frac{11601}{7840} e f_{z+1} - \frac{45}{112} e f_{z+2} - \frac{689}{560} e f_{z+3} + \frac{1017}{1120} e f_{z+4} - \frac{464}{735} e f_{z+\frac{9}{2}} + \frac{153}{1120} e f_{z+5}$$

$$y_{z+1} = y_{z+3} + \frac{53}{5670} e f_z - \frac{808}{2205} e f_{z+1} - \frac{409}{315} e f_{z+2} - \frac{307}{945} e f_{z+3} - \frac{43}{630} e f_{z+4} + \frac{256}{3969} e f_{z+\frac{9}{2}} - \frac{1}{63} e f_{z+5}$$

$$y_{z+2} = y_{z+3} - \frac{311}{90720} e f_z + \frac{2647}{70560} e f_{z+1} - \frac{485}{1008} e f_{z+2} - \frac{10331}{15120} e f_{z+3} + \frac{2561}{10080} e f_{z+4} - \frac{3056}{19845} e f_{z+\frac{9}{2}} + \frac{61}{2016} e f_{z+5}$$

$$y_{z+4} = y_{z+3} - \frac{151}{90720} e f_z + \frac{1063}{70560} e f_{z+1} - \frac{361}{5040} e f_{z+2} + \frac{8021}{15120} e f_{z+3} + \frac{7169}{10080} e f_{z+4} - \frac{4336}{19845} e f_{z+\frac{9}{2}} + \frac{353}{10080} e f_{z+5}$$

$$y_{z+\frac{9}{2}} = y_{z+3} - \frac{151}{107520} e f_z + \frac{3231}{250880} e f_{z+1} - \frac{225}{3584} e f_{z+2} + \frac{9029}{17920} e f_{z+3} + \frac{35703}{35840} e f_{z+4} + \frac{22}{735} e f_{z+\frac{9}{2}} + \frac{153}{7168} e f_{z+5}$$

$$y_{z+5} = y_{z+3} - \frac{11}{5670} e f_z + \frac{38}{2205} e f_{z+1} - \frac{5}{63} e f_{z+2} + \frac{517}{945} e f_{z+3} + \frac{533}{630} e f_{z+4} + \frac{9472}{19845} e f_{z+\frac{9}{2}} + \frac{61}{315} e f_{z+5} \quad (20)$$

Convergence analysis

Here, the convergence analysis of (14), (17) and (20) shall be worked-out.

Order and Error Constant

In [18], the Linear Multistep Method is said to be of order d if $c_0 = c_1 = 0, ..., c_p = 0$ but $c_{p+1} \neq 0$ and c_{p+1} is called the error constant. The order and error constants for (14) are obtained as follows

$$c_0 = c_1 = c_2 = c_3 = c_4 = c_5 = c_6 = c_7 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}^T \text{ but}$$

$$c_8 = \left(-\frac{3751}{3386880}, -\frac{89}{1128960}, -\frac{872075}{11098128384}, -\frac{79}{1003520}, -\frac{5929}{75497472}, -\frac{67}{846720} \right)^T$$

Therefore, (14) has order $p = 7$ and the error constant is

$$-\frac{3751}{3386880}, -\frac{89}{1128960}, -\frac{872075}{11098128384}, -\frac{79}{1003520}, -\frac{5929}{75497472}, -\frac{67}{846720}$$

Applying the same approach to (17), we obtained

$$c_0 = c_1 = c_2 = c_3 = c_4 = c_5 = c_6 = c_7 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}^T \text{ but}$$

$$c_8 = \left(-\frac{787}{211680}, \frac{317}{376320}, \frac{773}{3386880}, \frac{1721}{8028160}, \frac{244853}{1132462080}, \frac{1}{4704} \right)^T$$

Therefore, (17) has order $p = 7$ and the error constant is

$$-\frac{787}{211680}, \frac{317}{376320}, \frac{773}{3386880}, \frac{1721}{8028160}, \frac{244853}{1132462080}, \frac{1}{4704}$$

Applying the same approach to (20), we obtained

$$c_0 = c_1 = c_2 = c_3 = c_4 = c_5 = c_6 = c_7 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}^T \text{ but}$$

$$c_8 = \left(-\frac{57}{7840}, \frac{11}{10584}, -\frac{191}{211680}, -\frac{131}{211680}, -\frac{4017}{8028160}, -\frac{41}{52920} \right)^T$$

Therefore, (20) has order $p = 7$ and the error constants is

$$-\frac{57}{7840}, \frac{11}{10584}, -\frac{191}{211680}, -\frac{131}{211680}, -\frac{4017}{8028160}, -\frac{41}{52920}$$

Consistency

According to [18], a numerical method is said to be consistent if the order d is greater than 1 i.e. $p \geq 1$. Since the schemes in (14), (17) and (20) satisfy the condition for consistency of order $p \geq 1$, then the method is consistent.

Stability Analysis

In [19], a numerical method is said to be zero stable if the roots $\mu_{a,} a = 1, 2, 3, ..., n$ of the first characteristic polynomial $\pi(\mu)$ expressed as $\pi(\mu) = \det(\mu H_2^{(1)} - H_1^{(1)})$ satisfies $|\mu_a| \leq 1$ and the roots $|\mu_a|$ is simple or distinct.

The zero stability for (14) is estimated as

$$\begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_{z+1} \\ y_{z+2} \\ y_{z+\frac{9}{4}} \\ y_{z+\frac{5}{2}} \\ y_{z+\frac{11}{4}} \\ y_{z+3} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} y_{z-\frac{11}{4}} \\ y_{z-\frac{5}{2}} \\ y_{z-\frac{9}{4}} \\ y_{z-2} \\ y_{z-1} \\ y_{z} \end{pmatrix}$$

$$+e\begin{pmatrix} -\frac{6037}{3528} & \frac{47129}{2520} & -\frac{130048}{2835} & \frac{14296}{315} & -\frac{510464}{24255} & \frac{28817}{7560} \\ \frac{543}{1960} & \frac{3307}{840} & -\frac{4096}{567} & \frac{216}{35} & -\frac{20992}{8085} & \frac{3313}{7560} \\ \frac{250055}{903168} & \frac{520375}{129024} & -\frac{63755}{9072} & \frac{12325}{2016} & -\frac{200125}{77616} & \frac{168625}{387072} \\ \frac{1737}{6272} & \frac{18021}{4480} & -\frac{724}{105} & \frac{219}{35} & -\frac{636}{245} & \frac{1961}{4480} \\ \frac{567}{2048} & \frac{123823}{30720} & -\frac{44933}{6480} & \frac{1029}{160} & -\frac{6559}{2640} & \frac{119707}{276480} \\ \frac{611}{2205} & \frac{1264}{315} & -\frac{19456}{2835} & \frac{1984}{315} & -\frac{54272}{24255} & \frac{487}{945} \end{pmatrix} \begin{pmatrix} f_{z+1} \\ f_{z+2} \\ f_{z+\frac{9}{4}} \\ f_{z+\frac{5}{2}} \\ f_{z+\frac{11}{4}} \\ f_{z+3} \end{pmatrix} + e\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -\frac{67031}{249480} \\ 0 & 0 & 0 & 0 & 0 & -\frac{799}{249480} \\ 0 & 0 & 0 & 0 & 0 & -\frac{40825}{12773376} \\ 0 & 0 & 0 & 0 & 0 & -\frac{43}{13440} \\ 0 & 0 & 0 & 0 & 0 & -\frac{5831}{1824768} \\ 0 & 0 & 0 & 0 & 0 & -\frac{20}{6237} \end{pmatrix} \begin{pmatrix} f_{z-\frac{11}{4}} \\ f_{z-\frac{5}{2}} \\ f_{z-\frac{9}{4}} \\ f_{z-2} \\ f_{z-1} \\ f_{z} \end{pmatrix}$$

Where $H_2^{(1)} = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, H_1^{(1)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

and $D_2^{(1)} = \begin{pmatrix} -\frac{6037}{3528} & \frac{47129}{2520} & -\frac{130048}{2835} & \frac{14296}{315} & -\frac{510464}{24255} & \frac{28817}{7560} \\ \frac{543}{1960} & \frac{3307}{840} & -\frac{4096}{567} & \frac{216}{35} & -\frac{20992}{8085} & \frac{3313}{7560} \\ \frac{250055}{903168} & \frac{520375}{129024} & -\frac{63755}{9072} & \frac{12325}{2016} & -\frac{200125}{77616} & \frac{168625}{387072} \\ \frac{1737}{6272} & \frac{18021}{4480} & -\frac{724}{105} & \frac{219}{35} & -\frac{636}{245} & \frac{1961}{4480} \\ \frac{567}{2048} & \frac{123823}{30720} & -\frac{44933}{6480} & \frac{1029}{160} & -\frac{6559}{2640} & \frac{119707}{276480} \\ \frac{611}{2205} & \frac{1264}{315} & -\frac{19456}{2835} & \frac{1984}{315} & -\frac{54272}{24255} & \frac{487}{945} \end{pmatrix}$

$$\begin{aligned} \pi(\mu) &= \det\left(\mu H_2^{(1)} - H_1^{(1)}\right) \\ &= \left|\mu H_2^{(1)} - H_1^{(1)}\right| = 0. \end{aligned} \qquad (21)$$

We have,

$$\pi(\mu)=\left|\mu\begin{pmatrix}-1&0&0&0&0&0\\-1&1&0&0&0&0\\-1&0&1&0&0&0\\-1&0&0&1&0&0\\-1&0&0&0&1&0\\-1&0&0&0&0&1\end{pmatrix}-\begin{pmatrix}0&0&0&0&0&1\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix}\right|=\left|\begin{pmatrix}-\mu&0&0&0&0&0\\-\mu&\mu&0&0&0&0\\-\mu&0&\mu&0&0&0\\-\mu&0&0&\mu&0&0\\-\mu&0&0&0&\mu&0\\-\mu&0&0&0&0&\mu\end{pmatrix}-\begin{pmatrix}0&0&0&0&0&1\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix}\right|$$

$$\Rightarrow \pi(\mu)=\begin{pmatrix}-\mu&0&0&0&0&-1\\-\mu&\mu&0&0&0&0\\-\mu&0&\mu&0&0&0\\-\mu&0&0&\mu&0&0\\-\mu&0&0&0&\mu&0\\-\mu&0&0&0&0&\mu\end{pmatrix}.$$

Using Maple (18) software, we obtain

$$\pi(\mu)=-\mu^5(\mu+1)$$

$$\Rightarrow -\mu^5(\mu+1)=0$$

$\Rightarrow \mu_1=-1,\mu_2=0,\mu_3=0,\mu_4=0,\mu_5=0,\mu_6=0$ .Since $|\mu_i|<1,\ i=1,2,3,4,5,6$, (14) is zero stable.

Implementing the same approach, then **(**17**)** is presented as

$$\begin{pmatrix}0&-1&0&0&0&0\\1&-1&0&0&0&0\\0&-1&1&0&0&0\\0&-1&0&1&0&0\\0&-1&0&0&1&0\\0&-1&0&0&0&1\end{pmatrix}\begin{pmatrix}y_{z+1}\\y_{z+2}\\y_{z+3}\\y_{z+\frac{7}{2}}\\y_{z+\frac{15}{4}}\\y_{z+4}\end{pmatrix}=\begin{pmatrix}0&0&0&0&0&1\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix}\begin{pmatrix}y_{z-\frac{15}{4}}\\y_{z-\frac{7}{2}}\\y_{z-3}\\y_{z-2}\\y_{z-1}\\y_z\end{pmatrix}$$

$$+e\begin{pmatrix}-\frac{5506}{3465}&\frac{962}{2205}&-\frac{2582}{945}&\frac{14464}{2205}&-\frac{434176}{72765}&\frac{997}{630}\\-\frac{3461}{9240}&-\frac{5659}{5880}&\frac{9307}{7560}&-\frac{1864}{735}&\frac{159232}{72765}&-\frac{467}{840}\\-\frac{551}{27720}&\frac{6967}{17640}&\frac{8963}{7560}&-\frac{3032}{2205}&\frac{77312}{72765}&-\frac{629}{2520}\\-\frac{471}{24640}&\frac{24429}{62720}&\frac{391}{280}&-\frac{927}{980}&\frac{2416}{2695}&-\frac{3921}{17920}\\-\frac{77861}{4055040}&\frac{287567}{737280}&\frac{1536983}{1105920}&-\frac{9359}{11520}&\frac{1526}{1485}&-\frac{335111}{1474560}\\-\frac{2}{105}&-\frac{286}{735}&\frac{1322}{945}&-\frac{128}{147}&\frac{8192}{6615}&-\frac{29}{210}\end{pmatrix}\begin{pmatrix}f_{z+1}\\f_{z+2}\\f_{z+3}\\f_{z+\frac{7}{2}}\\f_{z+\frac{15}{4}}\\f_{z+4}\end{pmatrix}+e\begin{pmatrix}0&0&0&0&0&-\frac{3841}{13230}\\0&0&0&0&0&\frac{127}{13230}\\0&0&0&0&0&\frac{2}{1323}\\0&0&0&0&0&\frac{181}{125440}\\0&0&0&0&0&\frac{6419}{4423680}\\0&0&0&0&0&\frac{19}{13230}\end{pmatrix}\begin{pmatrix}f_{z-\frac{15}{4}}\\f_{z-\frac{7}{2}}\\f_{z-3}\\f_{z-2}\\f_{z-1}\\f_z\end{pmatrix}$$

where $H_2^{(2)}=\begin{pmatrix}0&-1&0&0&0&0\\1&-1&0&0&0&0\\0&-1&1&0&0&0\\0&-1&0&1&0&0\\0&-1&0&0&1&0\\0&-1&0&0&0&1\end{pmatrix},H_1^{(2)}=\begin{pmatrix}0&0&0&0&0&1\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix}$

and $D_2^{(2)} = \begin{pmatrix} -\frac{5506}{3465} & \frac{962}{2205} & -\frac{2582}{945} & \frac{14464}{2205} & -\frac{434176}{72765} & \frac{997}{630} \\ -\frac{3461}{9240} & -\frac{5659}{5880} & \frac{9307}{7560} & -\frac{1864}{735} & \frac{159232}{72765} & -\frac{467}{840} \\ -\frac{551}{27720} & \frac{6967}{17640} & \frac{8963}{7560} & -\frac{3032}{2205} & \frac{77312}{72765} & -\frac{629}{2520} \\ -\frac{471}{24640} & \frac{24429}{62720} & \frac{391}{280} & -\frac{927}{980} & \frac{2416}{2695} & -\frac{3921}{17920} \\ -\frac{77861}{4055040} & \frac{287567}{737280} & \frac{1536983}{1105920} & -\frac{9359}{11520} & \frac{1526}{1485} & -\frac{335111}{1474560} \\ -\frac{2}{105} & -\frac{286}{735} & \frac{1322}{945} & -\frac{128}{147} & \frac{8192}{6615} & -\frac{29}{210} \end{pmatrix}$

$$\pi(\mu) = \det\left(\mu H_2^{(2)} - H_1^{(2)}\right)$$
$$= \left|\mu H_2^{(2)} - H_1^{(2)}\right| = 0. \tag{22}$$

We have,

$$\pi(\mu) = \left|\mu \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}\right| = \left|\begin{pmatrix} 0 & -\mu & 0 & 0 & 0 & 0 \\ \mu & -\mu & 0 & 0 & 0 & 0 \\ 0 & -\mu & \mu & 0 & 0 & 0 \\ 0 & -\mu & 0 & \mu & 0 & 0 \\ 0 & -\mu & 0 & 0 & \mu & 0 \\ 0 & -\mu & 0 & 0 & 0 & \mu \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}\right|$$

$$\Rightarrow \pi(\mu) = \begin{vmatrix} 0 & -\mu & 0 & 0 & 0 & -1 \\ \mu & -\mu & 0 & 0 & 0 & 0 \\ 0 & -\mu & \mu & 0 & 0 & 0 \\ 0 & -\mu & 0 & \mu & 0 & 0 \\ 0 & -\mu & 0 & 0 & \mu & 0 \\ 0 & -\mu & 0 & 0 & 0 & \mu \end{vmatrix}.$$

Using Maple (18) software, we obtain:

$$\pi(\mu) = \mu^5(\mu+1)$$
$$\Rightarrow \mu^5(\mu+1) = 0$$

$\Rightarrow \mu_1 = -1, \mu_2 = 0, \mu_3 = 0, \mu_4 = 0, \mu_5 = 0, \mu_6 = 0$ .Since $|\mu_i| < 1,\ i = 1,2,3,4,5,6$, (17) is zero stable.

With the same procedure (20) can be presented as follows

$$\begin{pmatrix} 0 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} y_{z+1} \\ y_{z+2} \\ y_{z+3} \\ y_{z+4} \\ y_{z+\frac{9}{2}} \\ y_{z+5} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} y_{z-\frac{9}{2}} \\ y_{z-4} \\ y_{z-3} \\ y_{v-2} \\ y_{z-1} \\ y_z \end{pmatrix}$$

$$+e\begin{pmatrix} -\frac{11601}{7840} & -\frac{45}{112} & -\frac{689}{560} & \frac{1017}{1120} & -\frac{464}{735} & \frac{153}{1120} \\ -\frac{808}{2205} & -\frac{409}{315} & -\frac{307}{945} & -\frac{43}{630} & \frac{256}{3969} & -\frac{1}{63} \\ \frac{2647}{70560} & -\frac{485}{1008} & -\frac{10331}{15120} & \frac{2561}{10080} & -\frac{3056}{19845} & \frac{61}{2016} \\ \frac{1063}{70560} & -\frac{361}{5040} & \frac{8021}{15120} & \frac{7169}{10080} & -\frac{4336}{19845} & \frac{353}{10080} \\ \frac{3231}{250880} & -\frac{225}{3584} & \frac{9029}{17920} & \frac{35703}{35840} & \frac{22}{735} & \frac{153}{7168} \\ \frac{38}{2205} & -\frac{5}{63} & \frac{517}{945} & \frac{533}{630} & \frac{9472}{19845} & \frac{61}{315} \end{pmatrix}\begin{pmatrix} f_{z+1} \\ f_{z+2} \\ f_{z+3} \\ f_{z+4} \\ f_{z+\frac{9}{2}} \\ f_{z+5} \end{pmatrix} + e\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -\frac{1013}{3360} \\ 0 & 0 & 0 & 0 & 0 & \frac{53}{5670} \\ 0 & 0 & 0 & 0 & 0 & -\frac{311}{90720} \\ 0 & 0 & 0 & 0 & 0 & -\frac{151}{90720} \\ 0 & 0 & 0 & 0 & 0 & -\frac{151}{107520} \\ 0 & 0 & 0 & 0 & 0 & -\frac{11}{5670} \end{pmatrix}\begin{pmatrix} f_{z-\frac{9}{2}} \\ f_{z-4} \\ f_{z-3} \\ f_{z-2} \\ f_{z-1} \\ f_{z} \end{pmatrix}$$

where $H_2^{(3)} = \begin{pmatrix} 0 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{pmatrix}, H_1^{(3)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

and $D_2^{(3)} = \begin{pmatrix} -\frac{11601}{7840} & -\frac{45}{112} & -\frac{689}{560} & \frac{1017}{1120} & -\frac{464}{735} & \frac{153}{1120} \\ -\frac{808}{2205} & -\frac{409}{315} & -\frac{307}{945} & -\frac{43}{630} & \frac{256}{3969} & -\frac{1}{63} \\ \frac{2647}{70560} & -\frac{485}{1008} & -\frac{10331}{15120} & \frac{2561}{10080} & -\frac{3056}{19845} & \frac{61}{2016} \\ \frac{1063}{70560} & -\frac{361}{5040} & \frac{8021}{15120} & \frac{7169}{10080} & -\frac{4336}{19845} & \frac{353}{10080} \\ \frac{3231}{250880} & -\frac{225}{3584} & \frac{9029}{17920} & \frac{35703}{35840} & \frac{22}{735} & \frac{153}{7168} \\ \frac{38}{2205} & -\frac{5}{63} & \frac{517}{945} & \frac{533}{630} & \frac{9472}{19845} & \frac{61}{315} \end{pmatrix}$

$$\begin{aligned} \pi(\mu) &= \det\left(\mu H_2^{(3)} - H_1^{(3)}\right) \\ &= \left|\mu H_2^{(3)} - H_1^{(3)}\right| = 0. \end{aligned} \tag{23}$$

We have,

$$\pi(\mu) = \left|\mu\begin{pmatrix} 0 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}\right| = \left|\begin{pmatrix} 0 & 0 & -\mu & 0 & 0 & 0 \\ \mu & 0 & -\mu & 0 & 0 & 0 \\ 0 & \mu & -\mu & 0 & 0 & 0 \\ 0 & 0 & -\mu & \mu & 0 & 0 \\ 0 & 0 & -\mu & 0 & \mu & 0 \\ 0 & 0 & -\mu & 0 & 0 & \mu \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}\right|$$

$$\Rightarrow \pi(\mu)=\begin{pmatrix} 0 & 0 & -\mu & 0 & 0 & -1 \\ \mu & 0 & -\mu & 0 & 0 & 0 \\ 0 & \mu & -\mu & 0 & 0 & 0 \\ 0 & 0 & -\mu & \mu & 0 & 0 \\ 0 & 0 & -\mu & 0 & \mu & 0 \\ 0 & 0 & -\mu & 0 & 0 & \mu \end{pmatrix}.$$

Using Maple (18) software, we obtain

$$\pi(\mu)=-\mu^5(\mu+1)$$

$$\Rightarrow -\mu^5(\mu+1)=0$$

$\Rightarrow \mu_1=-1, \mu_2=0, \mu_3=0, \mu_4=0, \mu_5=0, \mu_6=0$. Since $|\mu_i|<1,\ i=1,2,3,4,5,6$, (20) is zero stable.

Convergence

The necessary and sufficient condition for a linear multistep method to be convergent is that it must be consistent and zero stable as stated by [19]. Since (14), (17) and (20) are both consistent and zero stable, therefore, the method is convergent.

Region of Absolute Stability

The regions of absolute stability of the numerical methods for DDEs are considered. We considered finding the P - and Q -stability by applying (14), (17) and (20) to the DDEs of this form

$$\begin{aligned} y'(t) &= my(t) + ny(t-\tau), && t \ge t_0 \\ y(t) &= a(t), && t \le t_0 \end{aligned} \tag{24}$$

where $a(t)$ is the initial function, m, n are complex coefficients, $\tau = ze, z \in Z^+$, e is the step size and $z=\dfrac{\tau}{e}, z$ is a positive integer. Let $P_1 = em$ and $P_2 = en$, then

Making use of Maple 18 and MATLAB, the region of P - and Q - stability for (14), (17) and (20) are plotted and shown as Fig.1 to 6. The P -stability regions in Fig 1 to 3 lie inside the open-ended region while the Q -stability regions in Fig 4 to 6 lie inside the enclosed region as shown below

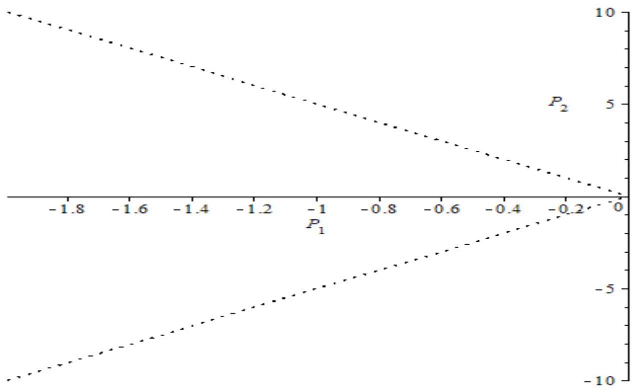

Fig.1.Region of P -stability (HEBAMM) in (14)

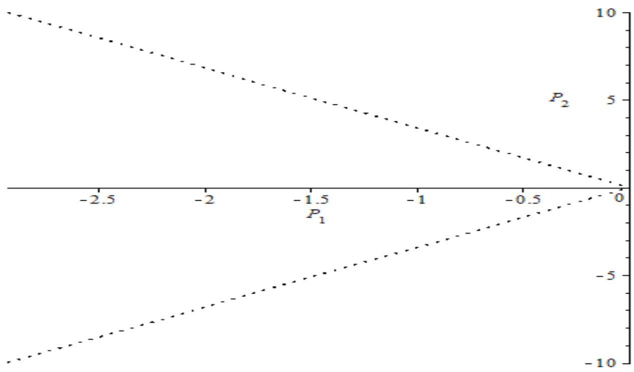

Fig.2.Region of P -stability (HEBAMM) in (17)

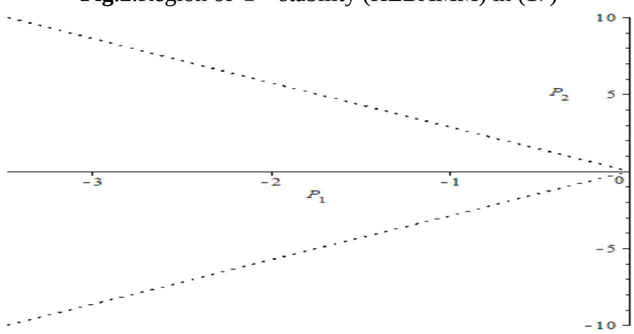

Fig.3.Region of P -stability (HEBAMM) in (20)

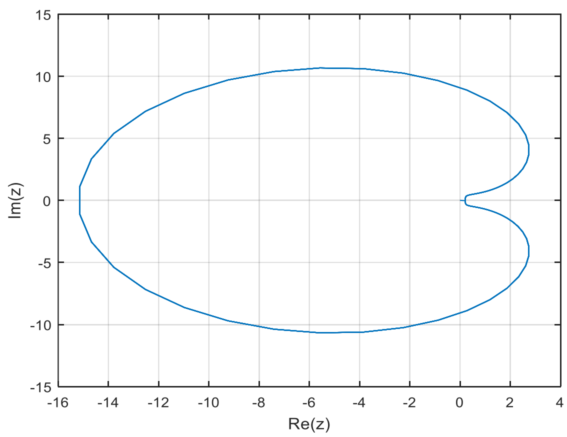

Fig.4.Region of Q -stability (HEBAMM) in (14)

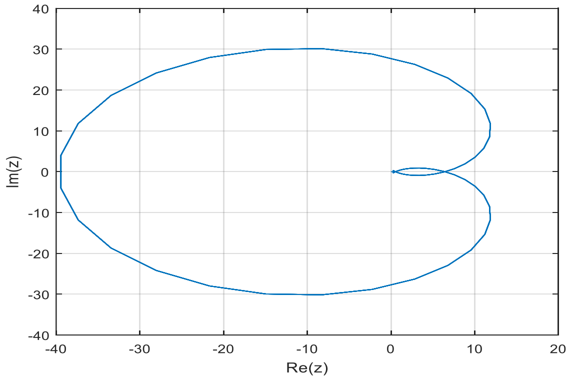

Fig.5.Region of Q -stability (HEBAMM) in (17)

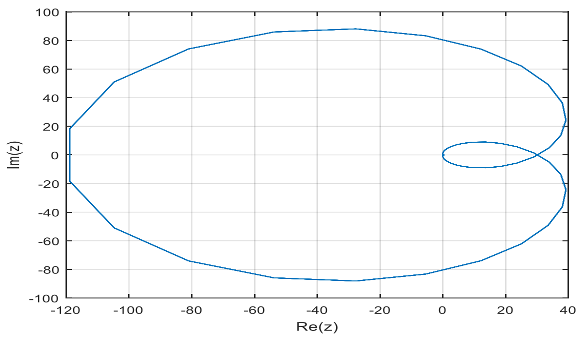

Fig.6.Region of Q -stability (HEBAMM) in (20)

Numerical Implementations

In this section, some first-order delay differential equations shall be solved using (14), (17) and (20) of the discrete schemes that have been established. The delay termshall be evaluated using the expression developed by [8].

Example 1

$$y'(t) = -1000y(t) + y(t-(\ln(1000-1))),$$
$$0 \le t \le 3$$
$$y(t) = e^{-t}, t \le 0$$

Exact solution $y(t) = e^{-t}$ in [8]

Example 2

$$y'(t) = -1000y(t) + 997e^{-3}y(t-1) + (1000-997e^{-3}),$$
$$0 \le t \le 3$$
$$y(t) = 1+e^{-3t}, t \le 0$$

Exact solution $y(t) = 1+e^{-3t}$ in [8]

Analysis and Comparison of Results

Here, the solutions of the schemes obtained in (14), (17) and (20), shall be investigated in solving the two examples above by estimating their absolute errors.

The results achieved after the application of the proposed method shall be compared to [9, 10, 13] to prove its superiority. The notations used in the table are stated below

HEABMM =Hybrid Extended Block Adams Moulton Methods for step numbers $k = 2, 3$ and 4.

RBBDFM = Reformulated Block Backward Differentiation Formulae Methods for step numbers $k = 3$ and 4 in [8].

TDBBDFM = Third Derivative Block Backward Differentiation Formulae Method for step numbers $k = 2, 3$ and 4 in [9].

CBBDFM = Conventional Block Backward Differentiation Formulae Method for step numbers $k = 2$ and 3 in [17].

MAXE = Maximum Error.

Table 4.1.1: Absolute Error of HEBAMM Method with Integrated Off-grid Points of $k = 2, 3$ and 4for Example 1

T	k = 2 Error	k = 3 Error	k = 4 Error
0.1	2.29627E-07	2.30E-08	2.30E-09
0.2	3.92953E-07	3.93E-08	3.93E-09
0.3	5.07362E-07	5.07E-08	5.07362E-09
0.4	5.86E-07	5.86E-08	5.85726E-09
0.5	6.37578E-08	6.38E-09	6.37578E-10
0.6	6.70E-08	6.70E-09	6.69971E-10
0.7	6.88E-08	6.88E-09	6.88128E-10
0.8	6.96E-08	6.96E-09	6.9591E-10
0.9	6.96E-08	6.96E-09	6.96174E-10
1	6.91E-08	6.91E-09	6.91031E-10
1.1	6.82E-09	6.82E-10	6.82E-11
1.2	6.07E-09	6.70E-10	6.70353E-11
1.3	6.57E-09	6.57E-10	6.56815E-11
1.4	6.42E-09	6.42E-10	6.42051E-11
1.5	6.27E-09	6.27E-10	6.26519E-11
1.6	6.11E-09	6.11E-10	6.10554E-11
1.7	5.94E-09	5.94E-10	5.94399E-11
1.8	5.78E-10	5.78E-11	5.78232E-12
1.9	5.62E-10	5.62E-11	5.62179E-12
2	5.46E-10	5.46E-11	5.46333E-12
2.1	5.31E-10	5.31E-11	5.30755E-12
2.2	5.15E-10	5.15E-11	5.15491E-12
2.3	5.01E-10	5.01E-12	5.00568E-13
2.4	4.86E-10	4.86E-12	4.86006E-13
2.5	4.72E-10	4.72E-12	4.71813E-13
2.6	4.58E-11	4.58E-12	4.57996E-13
2.7	4.45E-11	4.45E-13	4.44555E-14
2.8	4.31E-11	4.31E-13	4.31486E-14
2.9	4.19E-11	4.19E-13	4.18785E-14
3	4.06E-11	4.06E-13	4.06446E-14

Table 1: Absolute Error of HEBAMM Method with Integrated Off-grid Points of $k = 2, 3$ and 4for Example 2

t	k = 2 Error	k = 3 Error	k = 4 Error
0.1	8.52E-08	8.52124E-09	8.52124E-10
0.2	1.61468E-08	1.61468E-09	1.61468E-10
0.3	2.29627E-08	2.29627E-09	2.29627E-10
0.4	2.90469E-08	2.90469E-09	2.90E-10
0.5	3.44699E-09	3.44699E-10	3.45E-11
0.6	3.92953E-09	3.92953E-10	3.92953E-11
0.7	4.35809E-09	4.35809E-10	4.35809E-11
0.8	4.73787E-09	4.73787E-10	4.74E-11
0.9	5.07362E-09	5.07362E-10	5.07362E-11
1	5.36958E-09	5.36958E-10	5.36958E-11
1.1	5.62963E-10	5.62963E-11	5.62963E-12
1.2	5.86E-10	5.85726E-11	5.85726E-12
1.3	6.06E-10	6.06E-11	6.06E-12
1.4	6.23E-10	6.22761E-11	6.22761E-12
1.5	6.37578E-10	6.37578E-11	6.37578E-12
1.6	6.50E-10	6.50247E-11	6.50247E-12
1.7	6.60981E-10	6.60981E-11	6.60981E-12
1.8	6.70E-11	6.69971E-12	6.69971E-13
1.9	6.77391E-11	6.77391E-12	6.77391E-13
2	6.83395E-11	6.83395E-12	6.83395E-13
2.1	6.88128E-11	6.88128E-12	6.88128E-13
2.2	6.91716E-11	6.91716E-12	6.91716E-13
2.3	6.94E-12	6.94275E-13	6.94275E-14
2.4	6.96E-12	6.9591E-13	6.9591E-14
2.5	6.97E-12	6.96716E-13	6.97E-14
2.6	6.96778E-12	6.96778E-13	6.97E-14
2.7	6.96E-13	6.96174E-14	6.96E-15
2.8	6.95E-13	6.95E-14	6.95E-15
2.9	6.93E-14	6.93E-15	6.9324E-16
3	6.91E-14	6.91E-15	6.91E-16

Table 2: Comparison between the Maximum Absolute Errors of

HEBAMM $k = 2, 3$ and 4 with [8, 9,13] for constant step size d = 0.01 Using Example 1.

COMPUTATIONAL METHOD	COMPARED MAXEs WITH [8, 9, 13]
HEBAMM MAXE for k = 2	4.58E-11
HEBAMM MAXE for k = 3	4.45E-13
HEBAMM MAXE for k = 4	4.45E-14
RBBDFMAXE for k = 3	4.88E-06
RBBDF MAXE for k = 4	4.38E-06
TDBBDFM MAXE for k = 2	3.44E-03
TDBBDFM MAXE for k = 3	6.32E-03
TDBBDFM MAXE for k = 4	9.64E-03
CBBDF MAXE for k = 2	8.96E-05
CBBDF MAXE for k = 3	9.39E-06

CPU time of HEAMM for k = 2 is 0.242s, k = 3 is 0.223s and k = 4 is 0.208s

Table 3: Comparison between the Maximum Absolute Errors of EBAMM $k = 2, 3$ and 4 with [8, 9,13] for constant step size d = 0.01 Using Example 2.

COMPUTATIONAL METHOD	COMPARED MAXEs WITH [8, 9, 13]
HEBAMM MAXE for k = 2	6.93E-14
HEBAMM MAXE for k = 3	6.93E-15
HEBAMM MAXE for k = 4	6.93E-16
RBBDFMAXE for k = 3	1.54E-09
RBBDF MAXE for k = 4	1.04E-09
TDBBDFM MAXE for k = 2	3.44E-03
TDBBDFM MAXE for k = 3	6.29E-03
TDBBDFM MAXE for k = 4	9.64E-03
CBBDF MAXE for k = 2	6.32E-06
CBBDF MAXE for k = 3	5.10E-07

CPU time of HEAMM for k = 2 is 0.240s, k = 3 is 0.220s and k = 4 is 0.205s

Conclusions

The discrete schemes of (14), (17) and (20), were developed and were examined to be convergent, P- and Q-stable. Also, it was noticed in Tables 1,2, 3 and .4 that the HEBAMM for $k = 4$ scheme performed better than the HEBAMM schemes for step numbers $k = 3$ and $k = 2$ when compared with other existing methods. It is recommended that the HEBAMM schemes of higher step numbers perform better than the HEBAMM schemes of lower step numbers and also the step numbers of $k = 2, 3$ and $k = 4$ are suitable for solving DDEs. Further studies should be carried-out for step numbers $k = 5, 6, 7...$ on the derivations of discrete schemes of HEBAMM for numerical approximations of DDEs without the application interpolation formula in computing the delay term.

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