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A new Sixth-Order Runge - Kutta - Nyström Method for Special Second Order Ordinary Differential Equations

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A B S T R A C T

this article, an explicit Runge-Kutta-Nyström method of sixth order for solving directly second-order ordinary differential equations is constructed. The stability property of the new method is discussed. Numerical illustrations are presented to show the efficiency of the new method by comparing it with other existing Runge-Kutta-Nyström and Runge-Kutta methods in the scientific literature.

رونج – كوتا – نيستروم من الرتبة السادسة لحل المعادلات التفاضلية العادية من الرتبة التانية

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الكلمات المفتاحية:
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طرق رونج-كوتا-نيستروم .

الملخص

في هذا البحث تم اشتقاق طريقة رونج – كوتا – نيستروم من الرتبة السادسة لحل المعادلات التفاضلية العادية من الرتبة التانية. كذلك تم دراسة استقرار الطريقة و رسم منطقة الاستقرار لها. و لاظهار مدي كفاءة الطريقة لحل المعادلات تم تطبيقها علي مجموعة امثلة و مقارنة النتائج المتحصل عليها مع طرق سابقة من طرق رونج – كوتا – نيستروم و طرق رونج – كوتا.

1. Introduction

The numerical solution for ordinary differential equations (ODEs) have been the subject of research in numerical analysis. Various types of numerical methods have been proposed for solving ODEs.
In this article, consider the following initial-value problem (IVP) for a
second order differential equation of the form:

$$y'' = f(x, y), \qquad y(x_0) = y_0, \qquad y'(x_0) = y'_0 \qquad (1)$$

where $y \in \Re^n, f: \Re \times \Re^n \to \Re^n$.
This type of problems arise naturally in many applied science fields such as chemistry, molecular dynamics and quantum physics.
Conventionally, researchers has solved the problem (1) reducing it into a first-order system ODEs and then applying the suitable numerical methods to solve the resulting system, such numerical methods are Runge-Kutta methods (RK) or linear multistep methods. However, this technique is very expensive because more function evaluations need to be evaluated or computed, which leads to a longer execution time and computational effort (see [1-4]). Therefore, the direct numerical methods for solving higher order ODEs have attracted significant attention, because these direct methods demonstrated the efficiency in terms of accuracy and the number of function evaluations. The most common numerical methods for directly solving problem (1) is Runge-Kutta-Nyström method (RKN), which was developed by Nyström in 1925.
Furthermore, a lot of efforts by prominent scholars have been put on numerical methods for solving problem (1) (see [2, 7, 8, 9,10]).
The remainder of this paper is organized as follows: in Section 2, we construct a new explicit RKN method with sixth algebraic order to solve directly problem (1). In section 3, the linear stability of the

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constructed method is discussed. The number of tested problems to show the efficiency of the proposed method compared with the existing RKN and RK methods are given in the section 4.

2. Development a new RKN method

The general s-stage explicit RKN methods for solving second-order ODEs (1) can be defined as follows:(see[2])

$$y_{n+1} = y_n + hy'_n + h^2 \sum_{i=1}^{s} b_i\, k_i$$

$$y'_{n+1} = y'_n + h \sum_{i=1}^{s} d_i\, k_i \qquad (2)$$

$$k_i = f\left(x_n + c_i h, y_n + c_i h y'_n + h^2 \sum_{j=1}^{i-1} a_{ij}\, k_j\right)$$

where b_i, c_i and a_{ij} are real numbers, which can be expressed by using Butcher notation in the tableau as follows:

Table 1: Butcher tableau for RKN method

$$\begin{array}{c|cccc} 0 & 0 & & & \\ c_2 & a_{21} & 0 & & \\ .. & .. & .. & 0 & \\ c_s & a_{s1} & .. & a_{ss} & 0 \\ \hline & b_1 & .. & .. & b_s \\ & d_1 & .. & .. & d_s \end{array}$$

The order conditions for special explicit RKN methods are given in the following Table 2 (see[6]).

Table 2: Order conditions for explicit RKN methods up to order six.

Orders	Conditions
1	$\sum_i d_i = 1$
2	$\sum_i b_i = \frac{1}{2}$
	$\sum_i d_i c_i = \frac{1}{2}$
3	$\sum_i b_i c_i = \frac{1}{6}$
	$\sum_i d_i c_i^2 = \frac{1}{3}$
4	$\sum_i b_i c_i^2 = \frac{1}{12}$
	$\sum_i d_i c_i^3 = \frac{1}{4}$
	$\sum_{ij} d_i a_{ij} c_j = \frac{1}{24}$
5	$\sum_i b_i c_i^3 = \frac{1}{20}$
	$\sum_{ij} b_i a_{ij} c_j = \frac{1}{120}$
	$\sum_i d_i c_i^4 = \frac{1}{5}$
	$\sum_{ij} d_i c_i a_{ij} c_j = \frac{1}{30}$
	$\sum_{ij} d_i a_{ij} c_j^2 = \frac{1}{60}$
6	$\sum_i b_i c_i^4 = \frac{1}{30}$
	$\sum_{ij} b_i c_i a_{ij} c_j = \frac{1}{180}$
	$\sum_{ij} b_i a_{ij} c_j^2 = \frac{1}{360}$
	$\sum_i d_i c_i^5 = \frac{1}{6}$
	$\sum_{ij} d_i c_i^2 a_{ij} c_j = \frac{1}{36}$
	$\sum_{ij} d_i c_i a_{ij} c_j^2 = \frac{1}{72}$
	$\sum_{ij} d_i a_{ij} c_j^3 = \frac{1}{120}$
	$\sum_{ijk} d_i a_{ij} a_{jk} c_k = \frac{1}{4}$

In addition to $Ae = \frac{c^2}{2}$, these assumptions greatly simplify the order conditions to those given in Table 2.

To derive the sixth order RKN method, we will use the algebraic order conditions up to order six in Table 2 with the simplifying assumption

$$a_{2,1} = \frac{c_2^2}{2}, a_{31} + a_{32} = \frac{c_3^2}{2}, a_{41} + a_{42} + a_{43} = \frac{c_4^2}{2},$$

$$a_{51} + a_{52} + a_{53} + a_{54} = \frac{c_5^2}{2}$$

These equations are solved by maple software to obtain the following results as presented in the Butcher tableau and the new method denoted as RKN6(5) as seen in Table 3.

Table 3: Butcher tableau for RKN6(5) method

$$\begin{array}{c|ccccc} 0 & 0 & & & & \\ \frac{1}{2}-\frac{\sqrt{5}}{10} & a_{21} & 0 & & & \\ \frac{1}{2}+\frac{\sqrt{5}}{10}. & a_{31} & a_{32} & 0 & & \\ 0 & 0 & a_{42} & a_{43} & 0 & \\ 1 & a_{51} & a_{52} & a_{53} & \frac{1}{2} & 0 \\ \hline 1 & 0 & b_2 & b_3 & \frac{1}{12} & 0 \\ & 0 & \frac{5}{12} & \frac{5}{12} & \frac{1}{12} & \frac{1}{12} \end{array}$$

where

$$a_{21} = \frac{3}{20} - \frac{\sqrt{5}}{20};\ a_{31} = \frac{-1}{60} + \frac{\sqrt{5}}{60}; a_{32} = \frac{1}{6} + \frac{\sqrt{5}}{30}; a_{42} = \frac{-1}{6} + \frac{\sqrt{5}}{15};$$
$$a_{43} = \frac{1}{6} - \frac{\sqrt{5}}{15}; a_{51} = \frac{-2}{3} + \frac{\sqrt{5}}{6}; a_{52} = \frac{1}{12} + \frac{\sqrt{5}}{60};$$
$$a_{53} = \frac{7}{12} - \frac{11\sqrt{5}}{60}; b_2 = \frac{5}{24} + \frac{\sqrt{5}}{24}; b_3 = \frac{5}{24} - \frac{\sqrt{5}}{24}.$$

3. Stability Analysis of the new method

In this section, the linear stability of the new method is analyzed. Consider the following test equation: (see [3, 6, 9])

$$y'' = -\lambda^2 y;\ \lambda > 0 \qquad (3)$$

Applying (3) to the RKN method produces the difference equations:

$$\begin{bmatrix} y_{n+1} \\ y'_{n+1} \end{bmatrix} = M(H) \begin{bmatrix} y_n \\ y'_n \end{bmatrix}$$

where $H = \lambda h$;

$$M(H) = \begin{bmatrix} 1 + Hb^T N^{-1} e & 1 + Hb^T N^{-1} c \\ Hd^T N^{-1} e & 1 + Hd^T N^{-1} c \end{bmatrix}; N^{-1} = I - HA;$$

$e = (1,1, \dots, 1)^T$; $M(H)$ is called stability matrix.

The stability function associated with this method is given by:

$$\emptyset(\xi, H) = det|\xi I - M(H)|$$

The stability region of RKN method is defined by:

$$S_R = \{(x;\ y): |\lambda_i(M)| < 1; i = 1,2\}$$

where λ_i are eigenvalues of $M(H)$.
The stability polynomial associated with RKN6(5) method is given by

$$\emptyset(\xi, H) = \xi^2 + \left(\frac{7}{21600}(\sqrt{5} - 1)H^4 - \frac{1}{360}H^3 - \frac{1}{12}H^2 - H - 2\right)\xi + \left(\frac{\sqrt{5}}{10800} + \frac{1}{4800}\right)H^4 - \left(\frac{\sqrt{5}}{129600} + \frac{1}{8100}\right)H^5$$

The stability region of the new method is depicted in the Figure 1.

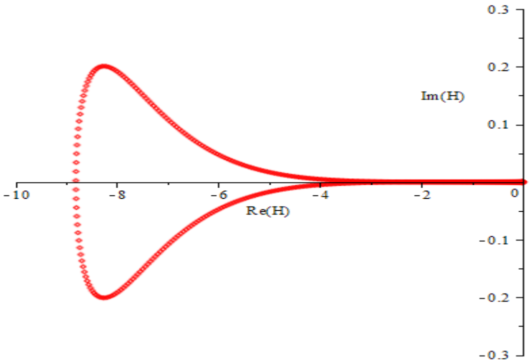

Fig. 1: Stability region for RKN6(5) method

4. Numerical experiments

In order to illustrate the efficiency of the method constructed in this paper, some problems are used for comparison. The criteria used in the numerical comparisons herein is based on computing the maximum error in the solution (Max Error= $max\,(|y(x_n) - y_n|)$) which is equal to the maximum between absolute errors of the exact solutions and the computed solutions. Figures 2–3 show the efficiency curves of $\log_{10}(\text{Max Error})$ against the number of function evaluations. The following methods are used in the comparison:

RKN6(5): The new sixth-order Runge-Kutta-Nyström method derived in this paper.

RKN6: The high order method of optimized embedded RKN 6(4) method derived by Anastassi and Kosti [5].

RK6: The sixth-order Runge-Kutta method constructed by Chan and Tsai [4].

Problem 1:

$$y'' = -y + 10\,e^{2x}(x), y(0) = 0, y'(0) = 0, 0 \le x \le 5$$

where the analytical solution is given by:

$$y = 2(e^{2x} - \cos(x) - 2\sin(x)).$$

Problem 2:

$$y'' = y + 10\sin^2(x), y(0) = -2, y'(0) = 0, 0 \le x \le 5$$

where the analytical solution is given by:

$$y = e^x + e^{-x} - 5 + \cos(2x)$$

Problem 3:

$$y'' = x^5 - y, y(0) = 0, y'(0) = 120, 0 \le x \le 5$$

where the analytical solution is given by:

$$y = x^5 - 20x^3 + 120x.$$

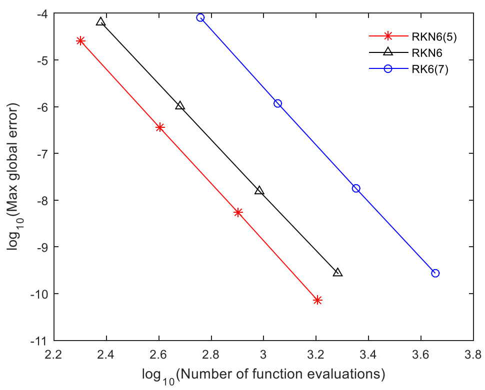

Fig. 2: The efficiency curves for Problem 1 with $h = \frac{1}{2^i}, i = 3, \ldots, 6$

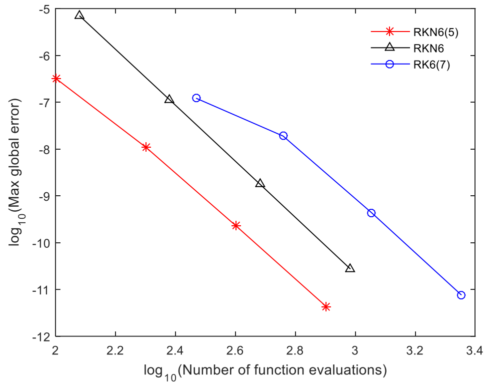

Fig. 3: The efficiency curves for Problem 2 with $h = \frac{1}{2^i}, i = 2, \ldots, 5$

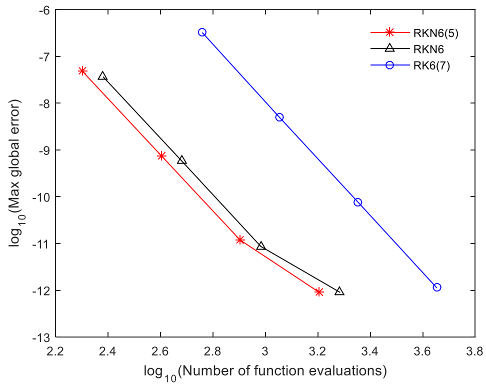

Fig. 3: The efficiency curves for Problem 3 with $h = \frac{1}{2^i}, i = 3, \ldots, 6$

5. Conclusion

In this study, a new RKN method of sixth order for the solution of special second order ODEs has been derived. The experimental results show that the function evaluations per step of the new method are less when compared with the other existing RKN and RK methods. Hence, the new method has less computational costs than the other existing methods therefore the efficiency of the developed method is higher than the other existing methods.

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