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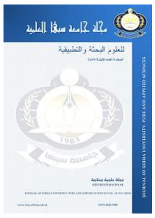

Application of Triple Shehu Transformation for solving Volterra Integro-Partial Differential Equation

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ABSTRACT
This paper depends on some concepts related to the triple Shehu transform; and how its various properties can be used to solve three-dimensional linear Volterra Integro-Partial Differential Equations; in terms of obtaining the exact solutions. Some examples will be studied to support the effectiveness of this transform.

تطبيق تحويل شهاو الثلاثي لحل معادلة فولتيرا التفاضلية الجزئية

سميحة الكليلي

قسم الرياضيات، كلية الأداب والعلوم المرج، جامعة بنغازي، ليبيا

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الملخص
تعتمد هذه الورقة على بعض المفاهيم المتعلقة بتحويل شهاو الثلاثي؛ وكيف يمكن استخدام خصائصه المختلفة لحل معادلات فولتيرا التفاضلية الجزئية التكاملية ثلاثية الأبعاد؛ من حيث الحصول على الحلول الدقيقة. ستتم دراسة بعض الأمثلة لدعم فعالية هذا التحويل.

Introduction

In science and engineering, integral-differential equations are utilized to describe a wide range of circumstances. Volterra Integro, in particular, is a critical differential equation utilized in nuclear processes, circuit analysis, and nano hydrodynamics, among other things, where numerous studies have focused on the development and use of sophisticated and efficient methods for solving Volterra Integro-Partial Differential Equations, such as He's Homotopy Perturbation Technique [1], differential transforms[2],and the sinc-collocation approach[3].

Many authors have previously combined integral transforms with the Adomian-decomposition method to solve these types of equations, such as the Laplace transform[4], Natural transform[5], Shehu transform[6], and other transforms. These transformations are very useful and efficient methods for solving partial differential equations with initial and boundary conditions.

In recent years, Bhadane, P. R.[7], Mousa, A., & Elzaki, T. M. [8-9], and AYDIN, M., & PEKER, H. A. [10] have applied the triple Laplace transforms, the triple Sumudutransform, and triple Elzaki transform, to solve Volterra Integro-Differential Equations.

In 2021, Alkaleeli, et al.[11], introduced the concept of triple Shehu transform as a novel generalization of the single Shehu transform[12], and utilized it to solve certain fractional partial differential equations.

The main idea of this paper is to solve three-dimensional linear Volterra Integro-Partial Differential Equations (LVIDEs), which have the following form: [13-14]

$$\frac{\partial^3 \psi(x,y,z)}{\partial x\,\partial y\,\partial z} + \psi(x,y,z) = f(x,y,z) + \int_{x_0}^{x}\int_{y_0}^{y}\int_{z_0}^{z} g\left(x,y,t,k,r,s,\psi(k,r,s)\right)dk\,dr\,ds \quad (2.1)$$

with respect to the initial conditions

$$\psi(0,0,0) = p_0,\ \psi(x,0,0) = p_1(x),\ \psi(0,0,z) = p_2(z),$$
$$\psi(0,y,0) = p_3(y),\ \psi(x,0,z) = p_4(x,z),\ \psi(x,y,0) = p_5(x,y),$$
$$\psi(0,y,z) = p_6(y,z), \quad (2.2)$$

where $\psi(x,y,z)$ is the unknown function, and the functions g and f are analytic in the domain of interest. By applying the triple Shehu transform method, we directly convert it into an algebraic equation, solve it, and then applying the inverse triple Shehu transform we obtain the exact solution. This method is demonstrated by presenting examples of various types that have already been discussed in [8].

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1. Preliminaries

This section provides some fundamental concepts and definitions of the triple Shehu transform, as well as several helpful characteristics that will be required throughout this study.

Definition 2.1 [12-15]
Over the set of the functions

$$A=\left\{ f(t): \exists M,\mu_1,\mu_2>0, \left|f(t)\right|<M e^{\left(\frac{|t|}{\mu_j}\right)}, \text{ if } t\in(-1)^j\times[0,\infty) \right\}$$

The Shehu Transform ($\mathcal{H}$) is defined by

$$\mathrm{H}(f(t))=F(s,u)=\int_0^\infty e^{\left(\frac{-st}{u}\right)} f(t)dt$$

$$=\lim_{\alpha\to\infty}\int_0^\alpha e^{\left(\frac{-st}{u}\right)} f(t)dt;\quad s,u>0. \qquad (2.3)$$

Definition 2.2[11-16] With regard to the variables *x, y,* and z, the single Shehu Transform ($\mathcal{H}$) of a function $f(x,y,z)$ are defined as:

$$\mathrm{H}_x\left(f(x,y,z)\right)=\int_0^\infty e^{-\left(\frac{sx}{u}\right)} f(x,y,z)dx \quad , (2.4)$$

$$\mathrm{H}_y\left(f(x,y,z)\right)=\int_0^\infty e^{-\left(\frac{qy}{v}\right)} f(x,y,z)dy \quad , (2.5)$$

$$\mathrm{H}_z\left(f(x,y,z)\right)=\int_0^\infty e^{-\left(\frac{rz}{k}\right)} f(x,y,z)dz \quad , (2.6)$$

Definition 2.3[11-16] The double Shehu Tranform (H^2) of a function $f(x,y,z)$ with respect to the variables *x, y, x,z* and *y,z* respectively, are defined as follows:

$$\mathrm{H}^2_{xy}\left(f(x,y,z)\right)=\int_0^\infty\int_0^\infty e^{-\left(\frac{sx}{u}+\frac{qy}{v}\right)} f(x,y,z)dx\,dy \quad , \qquad (2.7)$$

$$\mathrm{H}^2_{xz}\left(f(x,y,z)\right)=\int_0^\infty\int_0^\infty e^{-\left(\frac{sx}{u}+\frac{rz}{k}\right)} f(x,y,z)\,dxdz \quad , \qquad (2.8)$$

$$\mathrm{H}^2_{yz}\left(f(x,y,z)\right)=\int_0^\infty\int_0^\infty e^{-\left(\frac{qy}{v}+\frac{rz}{k}\right)} f(x,y,z)\,dydz \qquad (2.9)$$

Definition 2.4[11] If f is a continuous function with three variables, then the triple Shehu transform

(TRST) of $f(x, y, z)$ is defined as

$$\mathrm{H}^3_{xyz}\left(f(x,y,z)\right)=F\left[(s,q,r),(u,v,k)\right]$$

$$=\int_0^\infty\int_0^\infty\int_0^\infty e^{-\left(\frac{sx}{u}+\frac{qy}{v}+\frac{rz}{k}\right)} f(x,y,z)dx\,dy\,dz\,, \qquad (2.10)$$

where $x,y,z\geq 0$ and s,q,r,u,v and k, are Shehu variables, Provided the integral exists.
In addition, the inverse triple Shehu transform is defined by

$$\mathrm{H}^{-3}_{xyz}\left(F[(s,q,r),(u,v,k)]\right)=f(x,y,z)$$

$$=\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}\frac{1}{u}e^{\left(\frac{sx}{u}\right)}\left[\frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty}\frac{1}{v}e^{\left(\frac{qy}{v}\right)}\left(\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{1}{k}e^{\left(\frac{rz}{v}\right)}\mathrm{H}_3\left(f(x,y,z)\right)dr\right)dq\right]ds \qquad (2.11)$$

Some Useful Properties and Theorems Of Triple Shehu Transform

i. Linearity Property [11]

Let $f(x,y,z), g(x,y,z)$ be two functions, with triple Shehu tranform $F\left[(s,q,r),(u,v,k)\right]$ and $G\left[(s,q,r),(u,v,k)\right]$

Then, for any constants, α,β, we have

$$\mathrm{H}^3_{xyz}\left(\alpha f(x,y,z)+\beta g(x,y,z)\right)=\alpha\,\mathrm{H}^3_{xyz}\left(f(x,y,z)\right)+\beta\,\mathrm{H}^3_{xyz}\left(g(x,y,z)\right).$$

ii. TRST for mixed third order partial derivative of a function of three variables[11]

$$\mathrm{H}^3_{xyz}\left(\frac{\partial^3 f(x,y,z)}{\partial x\,\partial y\,\partial z}\right)=\left(\frac{sqr}{uvk}\right)F\left((s,q,r),(u,v,k)\right)-\left(\frac{sq}{uv}\right)\mathrm{H}^2_{xy}\left(f(x,y,0)\right)$$

$$-\left(\frac{qr}{vk}\right)\mathrm{H}^2_{yz}\left(f(0,y,z)\right)-\left(\frac{sr}{uk}\right)\mathrm{H}^2_{xz}\left(f(x,0,z)\right)$$

$$+\left(\frac{s}{u}\right)\mathrm{H}_x\left(f(x,0,0)\right)+\left(\frac{r}{k}\right)\mathrm{H}_z\left(f(0,0,z)\right)$$

$$+\left(\frac{q}{v}\right)\mathrm{H}_y\left(f(0,y,0)\right)-f(0,0,0)\,. \qquad (2.12)$$

iii. Triple Shehu Transforms of Integral

Let $f(x,y,z)$ be a functions such that

$$\mathrm{H}^3_{xyz}\left(f(x,y,z)\right)=F\left[(s,q,r),(u,v,k)\right] \qquad (2.13)$$

and,

$$g(x,y,z)=\int_0^x\int_0^y\int_0^z f(t_1,t_2,t_3)dt_1\,dt_2\,dt_3.$$

Then, we have

$$\mathrm{H}^3_{xyz}\left(\int_0^x\int_0^y\int_0^z f(t_1,t_2,t_3)dt_1\,dt_2\,dt_3\right)=\left(\frac{uvk}{sqr}\right)F\left[(s,q,r),(u,v,k)\right].$$

Proof

Let $g(x,y,z)=\int_0^x\int_0^y\int_0^z f(t_1,t_2,t_3)dt_1\,dt_2\,dt_3.$ Then

$$g_{xyz}(x,y,z)=f(x,y,z) \qquad (2.14)$$

and $g(0,0,0)=0.$ Using Eq(2.13), we get

$$\mathrm{H}^3_{xyz}\left(g_{xyz}(x,y,z)\right)=\mathrm{H}^3_{xyz}\left(f(x,y,z)\right)$$

$$=F\left[(s,q,r),(u,v,k)\right] \qquad (2.15)$$

By using Eq(2.12), we get

$$\mathrm{H}^3_{xyz}\left(g_{xyz}(x,y,z)\right)=\left(\frac{sqr}{uvk}\right)G\left((s,q,r),(u,v,k)\right)$$

$$-\left(\frac{sq}{uv}\right)\mathrm{H}^2_{xy}\left(g(x,y,0)\right)$$

$$-\left(\frac{qr}{vk}\right)\mathrm{H}^2_{yz}\left(g(0,y,z)\right)-\left(\frac{sr}{uk}\right)\mathrm{H}^2_{xz}\left(g(x,0,z)\right)$$

$$+\left(\frac{s}{u}\right)\mathrm{H}_x\left(g(x,0,0)\right)+\left(\frac{r}{k}\right)\mathrm{H}_z\left(g(0,0,z)\right)$$

$$+\left(\frac{q}{v}\right)\mathrm{H}_y\left(g(0,y,0)\right)-g(0,0,0)\,. \qquad (2.16)$$

Using Eq(2.15) and Eq(2.16), we obtain

$$\left(\frac{sqr}{uvk}\right)G\left((s,q,r),(u,v,k)\right)=F\left[(s,q,r),(u,v,k)\right]$$

$$+\left(\frac{sq}{uv}\right)\mathrm{H}^2_{xy}\left(g(x,y,0)\right)$$

$$+\left(\frac{qr}{vk}\right)\mathrm{H}^2_{yz}\left(g(0,y,z)\right)+\left(\frac{sr}{uk}\right)\mathrm{H}^2_{xz}\left(g(x,0,z)\right)$$

$$-\left(\frac{s}{u}\right)\mathrm{H}_x\left(g(x,0,0)\right)-\left(\frac{r}{k}\right)\mathrm{H}_z\left(g(0,0,z)\right)$$

$$-\left(\frac{q}{v}\right)\mathrm{H}_y\left(g(0,y,0)\right)+g(0,0,0)\,. \qquad (2.17)$$

We have,

$$\mathrm{H}_x\left(g\left(x,0,0\right)\right)=\mathrm{H}_y\left(g\left(0,y,0\right)\right)=\mathrm{H}_z\left(g\left(0,0,z\right)\right)$$
$$=\mathrm{H}_{xy}^2\left(g\left(x,y,0\right)\right)=\mathrm{H}_{xz}^2\left(g\left(x,0,z\right)\right)=\mathrm{H}_{yz}^2\left(g\left(0,y,z\right)\right)=0.$$

Therefore,

$$G\left((s,q,r),(u,v,k)\right)=\left(\frac{uvk}{sqr}\right)F\left[(s,q,r),(u,v,k)\right].$$

Hence,

$$\mathrm{H}_{xyz}^3\left(\int_0^x\int_0^y\int_0^z f\left(t_1,t_2,t_3\right)dt_1\,dt_2\,dt_3\right)=\left(\frac{uvk}{sqr}\right)F\left[(s,q,r),(u,v,k)\right].$$

iiv. The Convolution Theorem[11].

Let $f\left(x,y,z\right),g\left(x,y,z\right)$ be of exponential order, with triple Shehu tranform $F\left[(s,q,r),(u,v,k)\right]$ and $G\left[(s,q,r),(u,v,k)\right]$

Then, the triple Shehu transform of their convolution is,

$$\mathrm{H}_{xyz}^3\left[(f***g)(x,y,z)\right]=\mathrm{H}_{xyz}^3\left[f\left(x,y,z\right)\right]\mathrm{H}_{xyz}^3\left[g\left(x,y,z\right)\right] \quad (2.18)$$

Where

$$(f***g)(x,y,z)=\int_0^x\int_0^y\int_0^z f\left(x-t_1,y-t_2,z-t_3\right)g\left(t_1,t_2,t_3\right)dt_1\,dt_2\,dt_3$$

Proof: For the proof see [11, p.10].

For additional information on the triple Shehu transform [see 11].

2. Existence and uniqueness of the solution of the three-dimensional LVIDEs.

For the existence and uniqueness of the solution of Eq(2.1), with the initial condition in Eq(2.2) (see 13,p. 4,5).

4. Applications

In this section, the triple Shehu transform will be illustrated by studying the following examples.

Example4.1 [8]

Consider the linear volterra integro-differential equation

$$\frac{\partial^3\psi(x,y,z)}{\partial x\,\partial y\,\partial z}+\psi(x,y,z)=x\cos z-\frac{x^2y\sin z}{2}+\int_0^x\int_0^y\int_0^z\psi\left(t_1,t_2,t_3\right)dt_1dt_2dt_3. \quad (4.1)$$

With

$$\begin{cases}\psi(x,y,0)=\psi(x,0,0)=x,\\ \psi(x,0,z)=x\cos z,\\ \psi(0,y,z)=\psi(0,y,0)=\psi(0,0,z)=\psi(0,0,0)=0.\end{cases} \quad (4.2)$$

Applying the triple Shehu transform on both sides of Eq(4.1), we have

$$\left[1+\left(\frac{sqr}{uvk}\right)\right]\overline{\psi}\left((s,q,r),(u,v,k)\right)=G\left((s,q,r),(u,v,k)\right)+\left(\frac{u}{s}\right)^2\left(\frac{v}{q}\right)\left(\frac{rk}{k^2+r^2}\right)-\left(\frac{u}{s}\right)^3\left(\frac{v}{q}\right)^2\left(\frac{k^2}{k^2+r^2}\right)+\left(\frac{uvk}{sqr}\right)\overline{\psi}\left[(s,q,r),(u,v,k)\right] \quad (4.3)$$

Where,

$$G\left((s,q,r),(u,v,k)\right)=\left(\frac{sq}{uv}\right)\mathrm{H}_{xy}^2\left(\psi(x,y,0)\right)+\left(\frac{qr}{vk}\right)\mathrm{H}_{yz}^2\left(\psi(0,y,z)\right)+\left(\frac{sr}{uk}\right)\mathrm{H}_{xz}^2\left(\psi(x,0,z)\right)-\left(\frac{s}{u}\right)\mathrm{H}_x\left(\psi(x,0,0)\right)-\left(\frac{r}{k}\right)\mathrm{H}_z\left(\psi(0,0,z)\right)-\left(\frac{q}{v}\right)\mathrm{H}_y\left(\psi(0,y,0)\right)+\psi(0,0,0).$$

Therefore,

$$\left[1+\left(\frac{sqr}{uvk}\right)-\left(\frac{uvk}{sqr}\right)\right]\overline{\psi}\left((s,q,r),(u,v,k)\right)=\left(\frac{u}{s}\right)\left(\frac{r^2}{r^2+k^2}\right)+\left(\frac{u}{s}\right)^2\left(\frac{v}{q}\right)\left(\frac{rk}{r^2+k^2}\right)-\left(\frac{u}{s}\right)^3\left(\frac{v}{q}\right)^2\left(\frac{k^2}{r^2+k^2}\right)$$
$$=\left(\frac{u}{s}\right)^2\left(\frac{v}{q}\right)\left(\frac{kr}{r^2+k^2}\right)\left[1+\left(\frac{sqr}{uvk}\right)-\left(\frac{uvk}{sqr}\right)\right].$$

Hence,

$$\overline{\psi}\left((s,q,r),(u,v,k)\right)=\left(\frac{u}{s}\right)^2\left(\frac{v}{q}\right)\left(\frac{kr}{r^2+k^2}\right) \quad (4.4)$$

Operating inverse triple Shehu on both sides of Eq (4.4), we obtain the exact

$$\psi(x,y,z)=x\cos z.$$

This agrees with the solution obtained by Mousa, A., and Elzaki, M. [8].

Example4.2

Consider the linear volterra integro-differential equation [8]

$$\frac{\partial^3\psi(x,y,z)}{\partial x\,\partial y\,\partial z}+\psi(x,y,z)=x+y+z+\frac{1}{2}\left(x^2y\,z+xy^2z+xyz^2\right)-\int_0^x\int_0^y\int_0^z\psi\left(t_1,t_2,t_3\right)dt_1dt_2dt_3. \quad (4.5)$$

With

$$\begin{cases}\psi(x,y,0)=x+y,\\ \psi(0,y,z)=y+z,\\ \psi(x,0,z)=x+z,\\ \psi(0,0,0)=0,\\ \psi(x,0,0)=x,\psi(0,y,0)=y,\psi(0,0,z)=z.\end{cases} \quad (4.6)$$

Applying the triple Shehu transform on both sides of Eq(4.5), we have

$$\left[1+\left(\frac{sqr}{uvk}\right)\right]\overline{\psi}\left((s,q,r),(u,v,k)\right)=G\left((s,q,r),(u,v,k)\right)+\left(\frac{u}{s}\right)^3\left(\frac{v}{q}\right)^2\left(\frac{k}{r}\right)^2+\left(\frac{u}{s}\right)^2\left(\frac{v}{q}\right)^3\left(\frac{k}{r}\right)^2+\left(\frac{u}{s}\right)^2\left(\frac{v}{q}\right)^2\left(\frac{k}{r}\right)^3+\left(\frac{u}{s}\right)^2\left(\frac{v}{q}\right)\left(\frac{k}{r}\right)+\left(\frac{u}{s}\right)\left(\frac{v}{q}\right)^2\left(\frac{k}{r}\right)+\left(\frac{u}{s}\right)\left(\frac{v}{q}\right)\left(\frac{k}{r}\right)^2-\left(\frac{uvk}{sqr}\right)\overline{\psi}\left[(s,q,r),(u,v,k)\right] \quad (4.7)$$

Where,

$$G\left((s,q,r),(u,v,k)\right)=\left(\frac{sq}{uv}\right)\mathrm{H}_{xy}^2\left(\psi\left(x,y,0\right)\right)+\left(\frac{qr}{vk}\right)\mathrm{H}_{yz}^2\left(\psi\left(0,y,z\right)\right)+\left(\frac{sr}{uk}\right)\mathrm{H}_{xz}^2\left(\psi\left(x,0,z\right)\right)-\left(\frac{s}{u}\right)\mathrm{H}_x\left(\psi\left(x,0,0\right)\right)-\left(\frac{r}{k}\right)\mathrm{H}_z\left(\psi\left(0,0,z\right)\right)-\left(\frac{q}{v}\right)\mathrm{H}_y\left(\psi\left(0,y,0\right)\right)+\psi\left(0,0,0\right).$$

Hence,

$$\overline{\psi}\left((s,q,r),(u,v,k)\right)=\left(\frac{u}{s}\right)^2\left(\frac{v}{q}\right)\left(\frac{k}{r}\right)+\left(\frac{u}{s}\right)\left(\frac{v}{q}\right)^2\left(\frac{k}{r}\right)+\left(\frac{u}{s}\right)\left(\frac{v}{q}\right)\left(\frac{k}{r}\right)^2. \quad (4.8)$$

Operating inverse triple Shehu on both sides of Eq (4.8), we obtain the exact solution

$$\psi(x,y,z) = x+y+z \ .$$

This agrees with the solution obtained by Mousa, A., and Elzaki, M. [8].

Table 1: Triple Shehu transform for some functions [11]

$f(x,y,z)$	$F\left[(s,q,r),(u,v,k)\right]$
abc	$abc\left(\frac{uvk}{sqr}\right)$
$x\ y\ z$	$\left(\frac{uvk}{sqr}\right)^2$
$e^{ax+by+cz}$	$\frac{ukv}{(s-au)(q-bv)(r-kc)}$
$\cos(ax+by+cz)$	$\frac{ukv\,(rsq-abuvr-sbkvc-auqkc)}{(s^2+a^2u^2)(q^2+b^2v^2)(r^2+k^2c^2)}$
$\sin(ax+by+cz)$	$\frac{ukv\,(sqkc-abuvkc+sbvr+auqr)}{(s^2+a^2u^2)(q^2+b^2v^2)(r^2+k^2c^2)}$
$\cos x\ \cos y\ \cos z$	$\left(\frac{su}{s^2+u^2}\right)\left(\frac{qv}{q^2+v^2}\right)\left(\frac{rk}{r^2+k^2}\right)$
$\sin x\ \sin y\ \sin z$	$\left(\frac{u^2}{s^2+u^2}\right)\left(\frac{v^2}{q^2+v^2}\right)\left(\frac{k^2}{r^2+k^2}\right)$
$x^n\ y^m\ z^p$	$n!\left(\frac{u}{s}\right)^{n+1} m!\left(\frac{v}{q}\right)^{m+1} p!\left(\frac{k}{r}\right)^{p+1},\ n,m,p=0,1,2,\ldots$
$x^{\alpha}\ y^{\beta}\ z^{\gamma}$	$\Gamma(\alpha+1)\left(\frac{u}{s}\right)^{\alpha+1}\Gamma(\beta+1)\left(\frac{v}{q}\right)^{\beta+1}\Gamma(\gamma+1)\left(\frac{k}{r}\right)^{\gamma+1},\ \alpha,\beta,\gamma\geq -1$

5. Conclusion

Integro-Partial Differential Equations(IPDEs) are of great interest to find their solution, especially the Volterra integro-Partial differential equation. This work deals with the definition of triple Shehu transform (TSHT) and its inverse. It presents and discusses some important theorems and properties using this new transformation to obtain exact solutions to the three-dimensional linear Volterra integro-Partial differential equation(LVIPDE). Under the initial conditions, the triple Shehu transform investigation was successful in finding the exact solutions, as the results are identical to those given in [8]. Triple Shehu transform applications show that the exact solution of LVIPDE obtained by using less computational operations and less time using TSHT. We conclude that the triple Shehu transform approach is effective in solving equations of the considered type . In addition, there are numerous potential future research directions in this area, the triple Shehu transform can be applied for solving other linear and non-linear integro-partial differential equations, as well as fractional integro-partial differential equations, that arise in science and engineering.

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