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Some Oscillation Theorems for Second Order Nonlinear and Nonhomogeneous Differential Equations with alternating coefficients

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Abstract Some oscillation criteria for solutions of a general ordinary differential of second order of the form

$$\left(r(t)\dot{x}(t)\right)^{\bullet}+q(t)\Phi(g(x(t)),r(t)\dot{x}(t)) = H\left(t,x(t)\right)$$

with alternating coefficients are given. Our results improve and extend some existing results in the literature. Some illustrative examples are given with its numerical solutions, which are computed using Runge Kutta method of fourth order.

Keywords: Alternating coefficients, Nonhomogeneous Equations, Oscillation Solutions, Runge- Kutta Methods.
AMO(MOS) Subject Classification: 34A 34, 34K 11.

بعض نظريات التذبذب للمعادلات التفاضلية غير الخطية وغير المتجانسة من الدرجة الثانية ذات المعاملات المتناوبة

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الملخص في هــــذه الورقة، نقــدم بعض مــــعايير التذبذب لحلول المــــعادلات التفاضلية العادية من الــــدرجة الثانية ذات المعاملات المتناوبة ذات الصورة العامة:

$$\left(r(t)\dot{x}(t)\right)^{\bullet}+q(t)\Phi(g(x(t)),r(t)\dot{x}(t)) = H\left(t,x(t)\right)$$

تعتبر نتائج التذبذب المعطاة في هذه الورقة تحسين وتوسيع لبعض النتائج السابقة. يوجد بعض الامثلة التوضيحية للنتائج المعطاة والتي تم حلها عدديا باستخدام طريقة رونج-كوتا من الرتبة الرابعة باستخدام برنامج الماتلاب، لنبين من خلال هذا الحل العددي صحة نتائجنا النظرية المعطاة.

الكلمات المفتاحية: المعاملات المتناوبة، المعادلات الغير .متجانسة، حلول متذبذبة، طريقة رنج_كوتا.

1. Introduction

In this paper, we consider the second order nonlinear ordinary differential equation (ODE) of the form

$$\left(r(t)\dot{x}(t)\right)^{\bullet}+q(t)\Phi\left(g(x(t)),r(t)\dot{x}(t)\right) = H\left(t,x(t)\right) \qquad (1.1)$$

Where r, q *and* $p:[t_0,\infty)\rightarrow$ R are continuous functions and $r(t)>0$ *for* $t\geq t_0>0$. g is a continuous function for $x\in(-\infty,\infty)$, continuously differentiable and satisfies $xg(x)>0$ and $g'(x)\geq k>0$ for all $x\neq 0$. The function Φ is a continuous function on RxR with $u\Phi(u,v)>0$ for all $u\neq 0$ and $\Phi(\lambda u,\lambda v)=\lambda\Phi(u,v)$ for any $\lambda\in(0,\infty)$ and H is a continuous function on $[t_0,\infty)\times$R with $H(t,x(t))/g(x(t))\leq p(t)$ for all $x\neq 0$ and $t\geq t_0$. Throughout this paper, we restrict our attention only to the solutions of the ODE (1.1) which exist on some ray $[t_0,\infty)$. Such solution of the equation (1.1) is said to be oscillatory if it has an infinite number of zeros, and otherwise it is said to be non-oscillatory.

The ODE (1.1) is called oscillatory if all its solutions are oscillatory, and otherwise it is called non-oscillatory. The problem of finding oscillation criteria for second order nonlinear ordinary differential equations has received a great attention of many authors; see for example to [1-15]. Kamenev [7] studied the equation

$$\ddot{x}(t)+q(t)x(t)=0 \qquad (1.2)$$

And proved that the condition

$$\lim_{t\to\infty}\sup\frac{1}{t^{n-1}}\int_{t_0}^{t}(t-s)^{n-1}q(s)\,ds=\infty,$$

for some integer $n\geq 3$, is sufficient for the oscillation of the ODE (1.2). Yan [15] proved that if

$$\lim_{t\to\infty}\sup\frac{1}{t^{n-1}}\int_{t_0}^{t}(t-s)^{n-1}q(s)\,ds<\infty,$$

for some integer $n\geq 3$ and there is a continuous function Ω on $[t_0,\infty)$ with

$$\int_{t_0}^{\infty}\Omega_+^2(s)\,ds=\infty,$$

where $\Omega_+(t)=\max\{\Omega(t),0\},\ t\geq t_0$ such that

$$\lim_{t\to\infty}\sup\frac{1}{t^{n-1}}\int_{t_0}^{t}(t-s)^{n-1}q(s)\,ds\geq\Omega(T),$$

for every $T\geq t_0$. Then every solution of the ODE (1.2) oscillates. Philos [11] improved Kamenev's result [7] as follows: He supposed that there exist continuous functions $H,h:D=\{(t,s):t\geq s\geq t_0\}\to \mathrm{R}$ such that $H(t,t)=0$ for $t\geq t_0$ and $H(t,s)>0$ for $t>s\geq t_0$. H has a continuous and non-positive partial derivative on D with respect to the second variable such that

$$-\frac{\partial}{\partial s}H(t,s)=h(t,s)\sqrt{H(t,s)}\quad for\ all\ (t,s)\in D.$$

Then, equation (1.2) is oscillatory if

$$\lim_{t\to\infty}\sup\frac{1}{H(t,t_0)}\int_{t_0}^{t}\left(H(t,s)q(s)-\frac{1}{4}h^2(t,s)\right)ds=\infty.$$

Also, Philos [11] extended and improved Yan's result [15] by proving that H and h as in above, moreover, supposed that

$$0<\inf_{s\geq t_0}\left[\lim_{t\to\infty}\inf\frac{H(t,s)}{H(t,t_0)}\right]\leq\infty,$$

$$\lim_{t\to\infty}\sup\frac{1}{H(t,t_0)}\int_{t_0}^{t}h^2(t,s)\,ds<\infty,$$

and assume that $\Omega(t)$ as in Yan's result [15] with

$$\int_{t_0}^{\infty}\Omega_+^2(s)ds=\infty.$$

Then, equation (1.2) is oscillatory if

$$\lim_{t\to\infty}\sup\frac{1}{H(t,T)}\int_{T}^{t}\left(H(t,s)q(s)-\frac{1}{4}h^2(t,s)\right)ds\geq\Omega(T)\ \ for\ every\ T\geq t_0.$$

In this paper, we continue in this direction the study of oscillatory properties of the ODE (1.1). The purpose of this paper is to improve and extend the above- mentioned results. Our results are more general than the previous results.

2. MAIN RESULTS

We state and prove here our oscillation theorems

Theorem2.1: Suppose that

(1) $\dfrac{1}{\Phi(1,v)}<\dfrac{1}{C_0},\ C_0>0,$

(2) $q(t)>0\ for\ all\ t\geq t_0.$

Moreover, assume that there exist a differentiable function $\rho:[t_0,\infty)\to(0,\infty)$ and the continuous functions $h,H:D=\{(t,s):t\geq s\geq t_0\}\to \mathrm{R}$, the H has a continuous and non-positive partial derivative on D with respect to the second variable such that

$$H(t,t)=0\quad for\ t\geq t_o,\ H(t,s)>0\quad for\ t>s\geq t_0.$$

$$-\frac{\partial}{\partial s}H(t,s)=h(t,s)\sqrt{H(t,s)}\quad \forall(t,s)\in D.$$

If

(3) $\displaystyle\lim_{t\to\infty}\sup\frac{1}{H(t,t_0)}\int_{t_0}^{t}r(s)\rho(s)\sigma^2(t,s)ds<\infty,$

where $\sigma(t,s)=\left[h(t,s)-\dfrac{\dot{\rho}(s)}{\rho(s)}\sqrt{H(t,s)}\right].$

(4)

$$\lim_{t\to\infty}\sup\frac{1}{H(t,t_0)}\int_{t_0}^{t}H(t,s)\rho(s)\big(C_0q(s)-p(s)\big)ds=\infty.$$

Then, every solution of the ODE (1.1) is oscillatory.

Proof

Without loss of generality, we assume that there exists a solution $x(t)$ of the ODE (1.1) such that $x(t)>0\ on\ [T,\infty)\ for\ some\ \ T\geq t_0\geq 0$. We define the function $\omega(t)$ as

$$\omega(t)=\frac{\rho(t)r(t)\dot{x}(t)}{g(x(t))},\ t\geq T$$

This and (1.1) imply

$$\dot{\omega}(t) \le \rho(t)p(t) - \rho(t)q(t)\Phi(1, v(t)) + \frac{\dot{\rho}(t)}{\rho(t)}\omega(t) - \frac{k}{\rho(t)r(t)}\omega^2(t),\ t \ge T$$

Where $v(t) = \omega(t)/\rho(t)$.

Then by condition (1), we have for all $t \ge T$

$$\dot{\omega}(t) \le \rho(t)p(t) - C_0\rho(t)q(t) + \frac{\dot{\rho}(t)}{\rho(t)}\omega(t) - \frac{k}{\rho(t)r(t)}\omega^2(t),\ \ t \ge T$$

Integrate the last inequality multiplied by $H(t,s)$ from *T* to *t*, we have

$$\int_T^t H(t,s)\rho(s)\big(C_0 q(s) - p(s)\big)ds \le H(t,T)\omega(T) - \int_T^t \left[-\frac{\partial}{\partial s}H(t,s)\right]\omega(s)ds + \int_T^t \frac{\dot{\rho}(s)}{\rho(s)}H(t,s)\omega(s)ds$$

$$-\int_T^t \frac{kH(t,s)}{\rho(s)r(s)}\omega^2(s)ds$$

$$\le H(t,T)\omega(T) - \int_T^t \left[\frac{kH(t,s)}{\rho(s)r(s)}\omega^2(s) + \sigma(t,s)\sqrt{H(t,s)}\omega(s)\right]ds.$$

Where $\sigma(t,s) = h(t,s) - \dfrac{\dot{\rho}(s)}{\rho(s)}\sqrt{H(t,s)}.$

Hence, we have

$$\int_T^t H(t,s)\rho(s)\big(C_0 q(s) - p(s)\big)ds \le H(t,T)\omega(T) - \int_T^t \left[\sqrt{\frac{KH(t,s)}{\rho(s)r(s)}}\omega(s) + \frac{1}{2}\sqrt{\frac{\rho(s)r(s)}{k}}\sigma(t,s)\right]^2 ds$$

$$+\int_T^t \frac{\rho(s)r(s)}{4k}\sigma^2(t,s)ds \qquad (2.1)$$

Then, for $t \ge T$, we have

$$\int_T^t H(t,s)\rho(s)\big(C_0 q(s) - p(s)\big)ds \le H(t,T)\omega(T) + \frac{1}{4k}\int_T^t r(s)\rho(s)\sigma^2(t,s)\,ds, t \ge T$$

Dividing the last inequality by $H(t,T)$, taking the limit superior as $t \to \infty$ and by condition (3), we obtain

$$\limsup_{t\to\infty}\frac{1}{H(t,T)}\int_T^t H(t,s)\,\rho(s)\big(C_0 q(s) - p(s)\big)ds \le \omega(T) + \frac{1}{4k}\limsup_{t\to\infty}\frac{1}{H(t,T)}\int_T^t r(s)\rho(s)\sigma^2(t,s)\,ds < \infty,$$

which contradicts the condition (4) in theorem2.1. Hence, the proof is completed.

Example2.1: Consider the differential equation

$$\left(\dot{x}(t)/t^2\right)^{\bullet} + t^5\left(x^3(t) + \frac{3x^9(t)}{4x^6(t) + \left(\dot{x}(t)/t^2\right)^2}\right) = \frac{x^3(t)\cos x(t)}{t^7},\ t > 0. \qquad (2.2)$$

We have

$r(t)=\frac{1}{t^2}$, $q(t)=t^5$, $g(x)=x^3$,

$\Phi(u,v)=u+\frac{3u^3}{4u^2+v^2}$ and

$\frac{H(t,x(t))}{g(x(t))}=\frac{\cos(x(t))}{t^7}\le\frac{1}{t^7}=p(t).$

Let $H(t,s)=(t-s)^2\ge 0\ \ \forall t\ge s\ge t_0>0,$ then, $\frac{\partial}{\partial s}H(t,s)=-2(t-s)$ *and thus* $h(t,s)=-2.$

Taking $\rho(t)=t^2$ such that

$$(1)\ \ \limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}r(s)\rho(s)\sigma^2(t,s)ds=\limsup_{t\to\infty}\frac{1}{H(t,T)}\int_{T}^{t}r(s)\rho(s)\left(h(t,s)-\frac{\dot\rho(s)}{\rho(s)}\sqrt{H(t,s)}\right)^2 ds$$

$$=\limsup_{t\to\infty}\frac{1}{(t-T)^2}\int_{T}^{t}\left(-2-\frac{2}{s}(t-s)\right)^2 ds$$

$$=\frac{4}{T}<\infty,$$

$$(2)\ \ \limsup_{t\to\infty}\frac{1}{H(t,T)}\int_{T}^{t}H(t,s)\ \rho(s)\big(C_0q(s)-p(s)\big)ds=\limsup_{t\to\infty}\frac{1}{(t-T)^2}\int_{T}^{t}(t-s)^2\left(C_0s^7-\frac{1}{s^5}\right)ds$$

$$=\infty.$$

All conditions of Theorem2.1 are satisfied, then, the given equation (2.2) is oscillatory. Also the numerical solutions of the given differential equation (2.2) are computed using the Runge Kutta method of the fourth order. We have

$$\ddot x(t)=f(t,x(t),\dot x(t))=x^3(t)\cos(x(t))-\left(x^3(t)+\frac{3x^9(t)}{4x^6(t)+\dot x^2(t)}\right)$$

with initial conditions $x(1)=-1,\ \dot x(1)=0.5$ on the chosen interval $[1,100]$ and finding values of the functions *r, q* and *f* where we consider $H(t,x)=f(t)\,l(x)$ at *t=1, n=500* and *h= 0.198.*

Table 1: Numerical solution of ODE (2.2)

K	t_k	$x(t_k)$
1	1	-1
2	1.198	-0.8435
3	1.396	-0.5933
4	1.594	-0.2797
5	1.792	0.0651
6	1.99	0.4025
7	2.188	0.6958
.	.	.
.	.	.
14	3.574	-0.0124
15	3.772	-0.3534
16	3.97	-0.6553
.	.	.
.	.	.
24	5.554	0.3033
25	5.752	0.6133
26	5.95	0.858

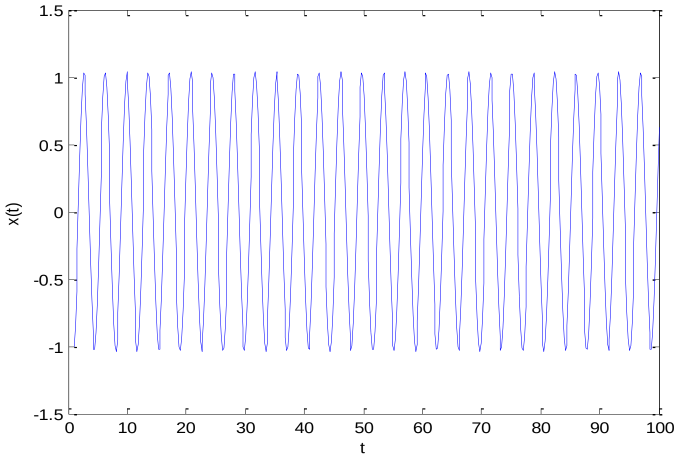

Figure1: Solution curve of ODE (2.2)

Theorem2.2

Suppose, in addition to the conditions (1), (2) and (3) in theorem 2.1 hold that there exist continuous functions h and H are defined as in Theorem2.1 and suppose that

(5) $$0 < \inf_{s \geq t_0}\left[\liminf_{t\to\infty} \frac{H(t,s)}{H(t,t_0)}\right] \leq \infty.$$

If there exists a continuous function Ω on $[t_0,\infty)$ such that

(6)

$$\limsup_{t\to\infty} \frac{1}{H(t,T)}\int_T^t \left[H(t,s)\rho(s)\left(C_0 q(s)-p(s)\right)-\frac{1}{4k}r(s)\rho(s)\sigma^2(t,s)\right]ds \geq \Omega(T)$$

for $T \geq t_0$, where $\sigma(t,s)=h(t,s)-\frac{\dot{\rho}(s)}{\rho(s)}\sqrt{H(t,s)}$, k is a positive constant and a differentiable function $\rho:[t_0,\infty)\to(0,\infty)$ and

(7) $$\int_T^\infty \frac{\Omega_+^2(s)}{\rho(s)r(s)}ds = \infty.$$

Where $\Omega_+(t)=\max\{\Omega(t),0\}$, then every solution of equation (1.1) is oscillatory.

Proof

Without loss of generality, we may assume that there exists a solution $x(t)$ of equation (1.1) such that $x(t) > 0$ on $[T,\infty)$ for some $T \geq t_0 \geq 0$. Dividing inequality (2.1) by $H(t,T)$ and taking the limit superior as $t\to\infty$, we obtain

$$\limsup_{t\to\infty} \frac{1}{H(t,T)}\int_T^t \left[H(t,s)\rho(s)\left(C_0 q(s)-p(s)\right)-\frac{1}{4k}\rho(s)r(s)\sigma^2(t,s)\right]ds \leq \omega(T)$$

$$-\limsup_{t\to\infty}\frac{1}{H(t,T)}\int_T^t\left[\sqrt{\frac{KH(t,s)}{\rho(s)r(s)}}\omega(s)+\frac{1}{2}\sqrt{\frac{\rho(s)r(s)}{k}}\sigma(t,s)\right]^2 ds$$

$$\leq \omega(T)-\liminf_{t\to\infty}\frac{1}{H(t,T)}\int_T^t\left[\sqrt{\frac{kH(t,s)}{\rho(s)r(s)}}\omega(s)+\frac{1}{2}\sqrt{\frac{\rho(s)r(s)}{k}}\sigma(t,s)\right]^2 ds$$

By condition (6), we get

$$\omega(T) \geq \Omega(T)+\liminf_{t\to\infty}\frac{1}{H(t,T)}\int_T^t\left[\sqrt{\frac{kH(t,s)}{\rho(s)r(s)}}\omega(s)+\frac{1}{2}\sqrt{\frac{\rho(s)r(s)}{k}}\sigma(t,s)\right]^2 ds$$

This shows that

$$\omega(T) \geq \Omega(T) \quad for\, every\, t \geq T, \tag{2.3}$$

and

$$\liminf_{t\to\infty} \frac{1}{H(t,T)}\int_T^t \left[\sqrt{\frac{kH(t,s)}{\rho(s)r(s)}}\omega(s)+\frac{1}{2}\sqrt{\frac{\rho(s)r(s)}{k}}\sigma(t,s)\right]^2 ds<\infty,$$

Hence,

$$\infty>\liminf_{t\to\infty} \frac{1}{H(t,t_0)}\int_{t_0}^t \left[\sqrt{\frac{kH(t,s)}{\rho(s)r(s)}}\omega(s)+\frac{1}{2}\sqrt{\frac{\rho(s)r(s)}{k}}\sigma(t,s)\right]^2 ds$$

$$\geq \liminf_{t\to\infty}\left[\frac{1}{H(t,t_0)}\int_{t_0}^t \frac{kH(t,s)}{\rho(s)r(s)}\omega^2(s)ds+\frac{1}{H(t,t_0)}\int_{t_0}^t \sigma(t,s)\sqrt{H(t,s)}\omega(s)ds\right] \quad (2.4)$$

Define

$$U(t)=\frac{1}{H(t,t_0)}\int_{t_0}^t \frac{kH(t,s)}{\rho(s)r(s)}\omega^2(s)\,ds,\, t\geq t_0$$

and

$$V(t)=\frac{1}{H(t,t_0)}\int_{t_0}^t \sigma(t,s)\sqrt{H(t,s)}\omega(s)ds,\ t\geq t_0.$$

Then, (2.4) becomes

$$\liminf_{t\to\infty}[U(t)+V(t)]<\infty. \quad (2.5)$$

Now, suppose that

$$\int_{t_0}^{\infty} \frac{\omega^2(s)}{\rho(s)r(s)}ds=\infty. \quad (2.6)$$

Then, by condition (5) we can easily see that

$$\lim_{t\to\infty} U(t)=\infty. \quad (2.7)$$

Let us consider a sequence $\{T_n\}_{n=1,2,3,\ldots}$ in $[t_0,\infty)$ *with* $\lim_{n\to\infty} T_n=\infty$ and such that

$$\lim_{n\to\infty}[U(T_n)+V(T_n)]=\liminf_{t\to\infty}[U(t)+V(t)].$$

By inequality (2.5) there exists a constant N such that

$$U(T_n)+V(T_n)\leq N,\ n=1,2,3,\ldots \quad (2.8)$$

From inequality (2.7), we have

$$\lim_{n\to\infty} U(T_n)=\infty. \quad (2.9)$$

And hence inequality (2.8) gives

$$\lim_{n\to\infty} V(T_n)=-\infty. \quad (2.10)$$

By taking into account inequality (2.9), from inequality (2.8), we obtain

$$1+\frac{V(T_n)}{U(T_n)}\leq\frac{N}{U(T_n)}<\frac{1}{2}.$$

Provided that n is sufficiently large. Thus

$$\frac{V(T_n)}{U(T_n)}<-\frac{1}{2},$$

which by inequality (2.10) and inequality (2.9) we have

$$\lim_{n\to\infty}\frac{V^2(T_n)}{U(T_n)}=\infty. \quad (2.11)$$

On the other hand by Schwarz's inequality, we have

$$V^2(T_n)=\frac{1}{H^2(T_n,t_0)}\left[\int_{t_0}^{T_n}\sigma(T_n,s)\sqrt{H(T_n,s)}\omega(s)ds\right]^2$$

$$\leq\left[\frac{1}{H(T_n,t_0)}\int_{t_0}^{T_n}\frac{\rho(s)r(s)}{k}\sigma^2(T_n,s)ds\right]\times\left[\frac{1}{H(T_n,t_0)}\int_{t_0}^{T_n}\frac{kH(T_n,s)}{r(s)\rho(s)}\omega^2(s)ds\right]$$

$$=\frac{1}{H(T_n,t_0)}\int_{t_0}^{T_n}\frac{\rho(s)r(s)}{k}\sigma^2(T_n,s)ds\times U(T_n).$$

Thus, we have

$$\frac{V^2(T_n)}{U(T_n)}\leq\frac{1}{H(T_n,t_0)}\int_{t_0}^{T_n}\frac{\rho(s)r(s)}{k}\sigma^2(T_n,s)ds \text{ for}$$

large n.

By inequality (2.11), we have

$$\frac{1}{k}\lim_{n\to\infty}\frac{1}{H(T_n,t_0)}\int_{t_0}^{T_n} r(s)\rho(s)\sigma^2(T_n,s)ds=\infty.$$

Consequently,

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^t r(s)\rho(s)\sigma^2(t,s)ds=\infty,$$

which contradicts to the condition (3) in Theorem2.1, Thus inequality (2.6) fails and hence

$$\int_{t_0}^{\infty}\frac{\omega^2(s)}{r(s)\rho(s)}ds<\infty.$$

Hence from inequality (2.3), we have

$$\int_{t_0}^{\infty}\frac{\Omega_+^2(s)}{r(s)\rho(s)}ds \le \int_{t_0}^{\infty}\frac{\omega^2(s)}{r(s)\rho(s)}ds < \infty,$$

which, contradicts to the condition (7), hence the proof is completed.

Example2.2

Consider the following differential equation

$$\left(\frac{\dot{x}(t)}{t^6}\right)^{\bullet} + \frac{1}{t^3}\left(x^7(t) + \frac{x^{133}(t)}{9x^{126}(t) + 6\left(\dot{x}(t)/t^6\right)^{18}}\right) = -\frac{x^9(t)\sin x(t)}{(x^2(t)+1)}, t > 0 \tag{2.12}$$

We note that $r(t) = \frac{1}{t^6}, q(t) = \frac{1}{t^3}, g(x) = x^7, \Phi(u,v) = u + \frac{u^{19}}{9u^{18} + 6v^{18}}$ *and*

$$\frac{H(t,x(t))}{g(x(t))} = -\frac{x^2(t)\sin x(t)}{(x^2(t)+1)} \le -\frac{x^2(t)}{(x^2(t)+1)} \le 0 = p(t) \quad for\, all\, t \ge t_0.$$

We let $H(t,s) = (t-s)^2 > 0 \quad for\, all\, t > s \ge t_0$, thus $\frac{\partial}{\partial s}H(t,s) = -2(t-s) = h(t,s)\sqrt{H(t,s)}$ for all $t \ge t_0 > 0$.

Taking $\rho(t) = 6$ such that

$$\limsup_{t\to\infty}\frac{1}{H(t,T)}\int_T^t r(s)\rho(s)\sigma^2(t,s)ds = \limsup_{t\to\infty}\frac{1}{H(t,T)}\int_T^t r(s)\rho(s)\left(h(t,s) - \frac{\dot{\rho}(s)}{\rho(s)}\sqrt{H(t,s)}\right)^2 ds$$

$$= \limsup_{t\to\infty}\frac{24}{(t-T)^2}\int_T^t \frac{1}{s^6}ds = 0 < \infty,$$

$$\inf_{s\ge t_0}\left[\liminf_{t\to\infty}\frac{H(t,s)}{H(t,t_0)}\right] = \inf_{s\ge t_0}\left(\liminf_{t\to\infty}\frac{(t-s)^2}{(t-t_0)^2}\right) == \inf_{s\ge t_0}(1) = 1,$$

$$thus\ 0 < \inf_{s\ge t_0}\left[\liminf_{t\to\infty}\frac{H(t,s)}{H(t,t_0)}\right] < \infty,$$

$$\limsup_{t\to\infty}\frac{1}{H(t,T)}\int_T^t\left[H(t,s)\rho(s)\left(C_0q(s) - p(s)\right) - \frac{r(s)\rho(s)}{4k}\sigma^2(t,s)\right]ds$$

$$= \limsup_{t\to\infty}\frac{1}{H(t,T)}\int_T^t\left[6C_0\frac{(t-s)^2}{s^3} - \frac{6}{ks^6}\right]ds = \frac{3C_0}{T^2} > \frac{3C_0}{4T^2}.$$

Set $\Omega(T) = \frac{3C_0}{4T^2}$, then $\Omega_+(T) = \frac{3C_0}{4T^2}$ and

$$\int_T^{\infty}\frac{\Omega^2{}_+(s)}{r(s)\rho(s)}ds = \frac{3C_0^2}{32}\int_T^{\infty}s^2ds = \infty.$$

All conditions of Theorem2.2 are satisfied, thus, the given equation (2.12) is oscillatory. We also compute the numerical solutions of the given differential equation (2.12) using the Runge Kutta method of fourth order (RK4). We have

$$\ddot{x}(t) = f(t, x(t), \dot{x}(t)) = \frac{-x^9(t)\sin x}{x^2(t)+1} - \left(x^7(t) + \frac{x^{133}(t)}{x^{126}(t) + 6\left(\dot{x}(t)\right)^{18}} \right)$$

with initial conditions $x(1) = -1,\ \dot{x}(1) = 0.5$ on the chosen interval $[1,100]$ and finding values of the functions r, *q* and *f* where we consider $H(t,x) = f(t)l(x)$ at *t=1*, *n=500* and *h=0.198.*

Table 2: Numerical solution of ODE (2.12)

K	t_k	$x(t_k)$
1	1	-1
2	1.198	-0.89
3	1.396	-0.7673
.	.	.
.	.	.
9	2.584	0.0081
10	2.782	0.1378
11	2.98	0.2674
.	.	.
.	.	.
28	6.346	-0.1154
29	6.544	-0.245
30	6.742	-0.3746
.	.	.
.	.	.
48	10.306	0.0579
49	10.504	0.187
50	10.702	0.316

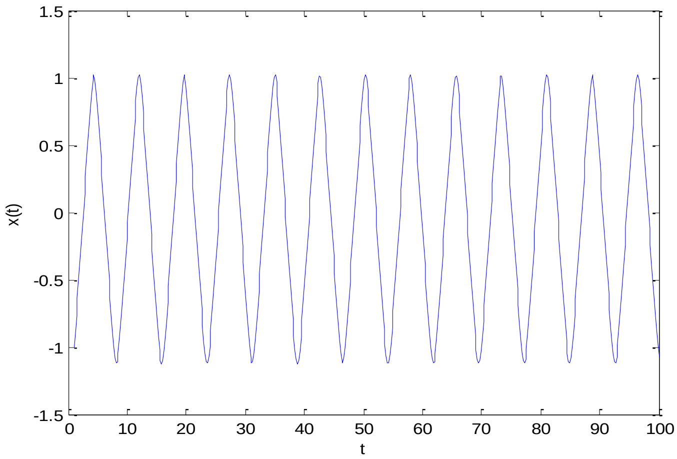

Figure2: Solution curve of ODE (2.12)

3. RESULTS AND DISCUSSOIN

Through this paper, some oscillation criteria for solutions of a general ordinary differential of second order of the form (1.1) are presented. Our results are more general than the previous results as follows:

1. Theorem2.1 extends Kamenev's result [7] and Philos's result [10] who studied a special case of the equation (1.1) as $r(t) \equiv 1$, $\Phi(g(x(t)), r(t)\dot{x}(t)) \equiv x(t)$ and $H(t, x(t)) \equiv 0.$
2. Results of Philos [10] and result of Kamenev [7] cannot applied to the given equation (2.2) in the example2.1.
3. Theorem2.2 extends and improves results of Philos [11] and results of Yan [15] who studied the equation (1.1) as $r(t) \equiv 1, \Phi(g(x(t)), r(t)\dot{x}(t)) \equiv x(t)$ *and* $H(t, x(t)) \equiv 0$.
4. In addition, results of Philos [11] and results of Yan [15] cannot be applied to the differential equation (2.12) in the example2.2.

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