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Some Results of Unique Common Fixed Point Generalized In Extended b_2-metric spaces

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ABSTRACT

Recently, extended b_2-metric space has been used as a generalization of both b_2-metric and extended b −metric spaces. The object of this study is to evidence some findings of fixed point and common fixed point for mappings in the frame of extended b_2-metric space, in addition, to generalizing some examples. The outcomes of this study are to generalize and extend some existing findings in previous literature.

بعض نتائج النقطة الثابتة المشتركة الوحيدة المعممة في الفراغات b_2 –المترية الموسعة

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الكلمات المفتاحية:
الفراغ b_2- المتري الموسع التام
راسم متصل
المتتالية المتقاربة
الفراغ b_2 - المتري الموسع.

الملخص

في الآونة الأخيرة، تم استخدام فراغ b_2-المتري الموسع كتعميم للفراغين b_2 - المتري و b- المتري الموسع. الهدف من هذه الدراسة، هو إثبات بعض نتائج النقطة الثابتة والنقطة الثابتة المشتركة لرواسم في الفراغ b_2- المتري الموسع، بالإضافة إلى تعميم بعض الأمثلة. تتمثل نتائج هذه الدراسة في تعميم وتوسيع بعض النتائج الموجودة في الأدبيات السابقة.

1. Introduction

There are two fundamental concepts in functional analysis, which stands the metric spaces and the fixed point.

Metric space is considered one of the essential topics that play a good role in the development of the study of other types of spaces, which contributed a distinguished part in the development of functional analysis theories.

One of the most significant applications of metric spaces is the investigation of theories of the existence and uniqueness of the fixed point, which is considered one of the essential topics in mathematics branches.

Furthermore, these theories play a significant role in demonstrating the existence of solutions to different types of equations.

Over the last four decades, there have been significant contributions to the development of the fixed-point theory, and widespread research has attended on the extension metric space extensions and the fixed points of several contractive type mappings. One of these generalizations is the 2-metric space was developed by S. G¨ahler [1] in the sixties as an extension of the metric space concept. Also, other extensions of metric space such as, b-metric space[2], extended b-metric space[3], and b_2-metric space[4]. In these novel extensions. Recently, the fourth condition of the definition of b2-metric space was then amended in 2018 by Elmabrok and Alkaleeli [5], yielding the extended b_2-metric space as a novel generalization of b_2-metric space. Various contraction-type conditions were, used to demonstrate some fixed point theorems (see [2-7], for instance). Concerning the structure of extended b_2-metric spaces, this study aims to extend and generalize some of the findings of Aage.,et.al.[8], which was previously generalized by Alkaleeli.,et.al.[9] in the frame of b_2-metric space.

2. Preliminaries

This section retraces the definitions of several types generalized metric spaces, along with some theorems and properties of extended b_2-metric spaces, that will be used later.

In the definitions that follow, Υ represents in for a non-empty set.

Definition2.1[1]. A 2−metric space is known as a pair

(Υ,), where $\rho: \Upsilon \times \Upsilon \times \Upsilon \to [0, \infty)$ is a function that fulfils the following requirements, for all $\chi, \tau, \nu \in \Upsilon$.

1. There is $\upsilon \in \Upsilon$, so that $\rho(\chi,\tau,\upsilon) \neq 0$

   for any $\chi \neq \tau \in \Upsilon$,

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2. If two or all of χ, τ and ν are equal, then

   $\rho(\chi,\tau,\upsilon) = 0$,

3. $\rho(\chi,\tau,\upsilon) = \rho(\chi,\upsilon,\tau) = \rho(\upsilon,\chi,\zeta) = \rho(\tau,\upsilon,\chi) = \rho(\upsilon,\chi,\tau) = \rho(\upsilon,\tau,\chi)$,
4. $\rho(\chi,\tau,\upsilon) \leq \rho(\chi,\zeta,\nu) + \rho(\tau,\zeta,\nu) + \rho(\zeta,\chi,\nu)$,

   for any $\zeta \in \Upsilon$.

Definition2.2[4] A b_2−metric space space is known as a pair (Υ, ρ), where $\rho: \Upsilon \times \Upsilon \times \Upsilon \rightarrow [0,\infty)$ is a function that fulfils the following requirements, for all $\chi, \tau, \nu \in \Upsilon$, and real number $s \geq 1$.

1. There is $\upsilon \in \Upsilon$, so that $\rho(\chi,\tau,\upsilon) \neq 0$

   for each $\chi \neq \tau \in \Upsilon$,

2. If two or all of χ, τ and ν are equal, then

   $\rho(\chi,\tau,\upsilon) = 0$,

3. $\rho(\chi,\tau,\upsilon) = \rho(\chi,\upsilon,\tau) = \rho(\upsilon,\chi,\zeta) = \rho(\tau,\upsilon,\chi) = \rho(\upsilon,\chi,\tau) = \rho(\upsilon,\tau,\chi)$,
4. $\rho(\chi,\tau,\upsilon) \leq s[\rho(\chi,\zeta,\nu) + \rho(\tau,\zeta,\nu) + \rho(\zeta,\chi,\nu)]$,

   for any $\zeta \in \Upsilon$.

Definition 2.3[5] Let $\varphi : \Upsilon \times \Upsilon \times \Upsilon \rightarrow [\,1,\infty)$ be a mapping. An extended b_2−metric space is known as a pair (Υ,ρ_φ), whrer $\rho_\varphi : \Upsilon \times \Upsilon \times \Upsilon \rightarrow [0,\infty)$ is a function that fulfils the following requirements for every $\chi, \tau, \nu \in \Upsilon$.

1. There is $\upsilon \in \Upsilon$, so that $\rho_\varphi(\chi,\tau,\upsilon) \neq 0$ for each $\chi \neq \tau \in \Upsilon$,

2. If two or all of χ, τ and $\upsilon$are equal, then $\rho_\varphi(\chi,\tau,\upsilon) = 0$,

3. $\rho_\varphi(\chi,\tau,\upsilon) = \rho_\varphi(\chi,\upsilon,\tau) = \rho_\varphi(\tau,\chi,\upsilon) = \rho_\varphi(\tau,\upsilon,\chi) = \rho_\varphi(\upsilon,\chi,\tau) = \rho_\varphi(\upsilon,\tau,\chi)$,

4. $\rho_\varphi(\chi,\tau,\upsilon) \leq \varphi(\chi,\zeta,\upsilon)[\rho_\varphi(\chi,\tau,\zeta) + \rho_\varphi(\chi,\upsilon,\zeta) + \rho_\varphi(\tau,\upsilon,\zeta)]$, for any $\zeta \in \Upsilon$.

Remarks2.1 [6]

From the above definition we can deduce that:

1. If $\varphi(\chi,\tau,\upsilon) = s \geq 1$, any extended b_2-metric space transforms into a b_2-metric space. In other words, an extended b_2-metric is more general than b_2 −metric(see [5-7]),
2. using condition (1), it readily verified that $\chi = \tau$, if $\rho_\varphi(\chi,\tau,\upsilon) = 0$ for every $\upsilon \in \Upsilon$.

We have generalized some Examples.

Example 2.1[10]

Let $\Upsilon =$ ‘ , define $\rho_\varphi: \Upsilon \times \Upsilon \times \Upsilon \longrightarrow [\,0,\infty\,)$, $\varphi: \Upsilon \times \Upsilon \times \Upsilon \rightarrow [\,1,\infty)$ by

$$\rho_\varphi(\chi,\tau,\upsilon) = \begin{cases} \chi^2+\tau^2+\upsilon^2, & \chi,\tau,\upsilon \in X, \ \chi \neq \tau \neq \upsilon, \\ 0 & \text{if at least two of } \chi,\tau \text{ and } \upsilon \text{ are equal.} \end{cases}$$

with

$$\varphi(\chi,\tau,\upsilon) = |\chi| + |\tau| + |\upsilon| + 1.$$

Obviously, (Υ,ρ_φ) is an extended b_2-metric space.
In fact, (1), (2), and (3) are so evident in Definition 2.3.
Therefore, we must still demonstrate that the fourth requirement of definition 2.3 is fulfilled for each $\chi, \tau, \nu, \zeta \in \Upsilon$.

i) If two or all of χ, τ and υ are equal, then the fourth requirement is clear,

ii) let $\chi \neq \tau \neq \upsilon$, will be acquired the following cases:

case.1 taking $\chi = \zeta$, we receive

$$\begin{aligned} \varphi(\chi,\tau,\upsilon)\left[\rho_\varphi(\chi,\tau,\zeta)+\rho_\varphi(\chi,\upsilon,\zeta)+\rho_\varphi(\tau,\upsilon,\zeta)\right] &= (1+|\chi|+|\tau|+|\upsilon|)\left[0+(\zeta^2+\tau^2+\upsilon^2)\right] \\ &> (\chi^2+\tau^2+\upsilon^2) \\ &= \rho_\varphi(\chi,\tau,\upsilon). \end{aligned}$$

The inequality is also valid in the situation of $\tau = \zeta, \upsilon = \zeta$.

case.2 taking $\chi \neq \zeta, \tau \neq \zeta, \upsilon \neq \zeta$, we receive

$$\begin{aligned} \varphi(\chi,\tau,\upsilon)\left[\rho_\varphi(\chi,\tau,\zeta)+\rho_\varphi(\chi,\upsilon,\zeta)+\rho_\varphi(\tau,\upsilon,\zeta)\right] &= (1+|\chi|+|\tau|+|\upsilon|)\left[2(\chi^2+\tau^2+\upsilon^2)+3\zeta^2\right] \\ &> (\chi^2+\tau^2+\upsilon^2) \\ &= \rho_\varphi(\chi,\tau,\upsilon). \end{aligned}$$

Given the preceding situations, it emerges that (4) is valid. As a consequence, the claim is valid.

Example2.2[4,11]

Let $\Upsilon = \left\{\frac{1}{4}, \frac{1}{4^2}, \ldots, \frac{1}{4^n}, \ldots\right\} \cup \{0,1\}$,

define $\rho_\varphi: \Upsilon \times \Upsilon \times \Upsilon \longrightarrow [\,0,\infty\,)$, $\varphi: \Upsilon \times \Upsilon \times \Upsilon \rightarrow [\,1,\infty)$ by

$$\rho_\varphi(\chi,\tau,\upsilon) = \begin{cases} (\chi\tau+\chi\upsilon+\tau\upsilon)^2, & \text{if } \chi \neq \tau \neq \upsilon, \\ 0, & \text{if at least two of } \chi,\tau \text{ and } \upsilon \text{ are equal.} \end{cases}$$

With,

$$\varphi(\chi,\tau,\upsilon) = \chi + \tau + \upsilon + 3.$$

Obviously, (Υ,ρ_φ) is an extended b_2-metric space.
In fact, (1), (2), and (3) are so evident in Definition 2.3.
Therefore, we must still demonstrate that the fourth requirement of definition 2.3 is fulfilled for each $\chi, \tau, \nu, \zeta \in \Upsilon$.

i) If two or all of χ, τ and υ are equal, then the fourth requirement is clear,

ii) let $\chi \neq \tau \neq \upsilon$, will be acquired the following cases:

case.1 taking $\chi = \zeta$, we obtain

$$\rho_\varphi(\chi,\tau,\zeta) = \rho_\varphi(\chi,\zeta,\upsilon) = 0.$$

Therefore,

$$\begin{aligned} \rho_\varphi(\chi,\tau,\upsilon) &= (\chi\tau+\chi\upsilon+\tau\upsilon)^2 \\ &< (\chi+\tau+\upsilon+3)\left[0+0+(\chi\tau+\chi\upsilon+\tau\upsilon)^2\right] \\ &= (\chi+\tau+\upsilon+3)\left[0+0+(\zeta\tau+\zeta\upsilon+\tau\upsilon)^2\right] \\ &= \varphi(\chi,\tau,\upsilon)\left[\rho_\varphi(\chi,\tau,\zeta)+\rho_\varphi(\chi,\zeta,\upsilon)+\rho_\varphi(\zeta,\tau,\upsilon)\right]. \end{aligned}$$

The inequality is also valid in the situation of $\tau = \zeta, \upsilon = \zeta$.

case.2 taking $\chi \neq \zeta, \tau \neq \zeta, \upsilon \neq \zeta$, we have

$$(\chi\tau+\chi\upsilon+\tau\upsilon) < (\chi\tau+\chi\zeta+\tau\zeta)+(\chi\zeta+\chi\upsilon+\zeta\upsilon)+(\zeta\tau+\zeta\upsilon+\tau\upsilon).$$

Consequently,

$$\begin{aligned} (\chi\tau+\chi\upsilon+\tau\upsilon)^2 &< \left[(\chi\tau+\chi\zeta+\tau\zeta)+(\chi\zeta+\chi\upsilon+\zeta\upsilon)+(\zeta\tau+\zeta\upsilon+\tau\upsilon)\right]^2, \\ &\leq 3\left[(\chi\tau+\chi\zeta+\tau\zeta)^2+(\chi\zeta+\chi\upsilon+\zeta\upsilon)^2+(\zeta\tau+\zeta\upsilon+\tau\upsilon)^2\right], \\ &\leq (\chi+\tau+\upsilon+3)\left[(\chi\tau+\chi\zeta+\tau\zeta)^2+(\chi\zeta+\chi\upsilon+\zeta\upsilon)^2+(\zeta\tau+\zeta\upsilon+\tau\upsilon)^2\right]. \end{aligned}$$

Accordingly,

$$\rho_\varphi(\chi,\tau,\upsilon) \leq \varphi(\chi,\tau,\upsilon)\left[\rho_\varphi(\chi,\tau,\zeta)+\rho_\varphi(\chi,\zeta,\upsilon)+\rho_\varphi(\zeta,\tau,\upsilon)\right].$$

Given the preceding situations, it emerges that (4) is valid. As a consequence, the claim is valid.

Example2.3[6,11]

Let $\Upsilon = l^\infty(\mathbb{R})$. The funtion $\rho_\varphi : \Upsilon \times \Upsilon \times \Upsilon \longrightarrow [0,\infty)$ described by

$$\rho_\varphi(\chi,\tau,\upsilon) = \left\{\sup_{r\in\mathbb{N}} \min\{|\chi_r - \tau_r|, |\chi_r - \upsilon_r|, |\tau_r - \upsilon_r|\}\right\}^p, p > 1,$$

with $\varphi : \Upsilon \times \Upsilon \times \Upsilon \longrightarrow [1,\infty)$ by

$$\varphi(\chi,\tau,\upsilon) = \sup_{r\in\mathbb{N}} \frac{|\chi_r + \tau_r + \upsilon_r|}{|\chi_r + \tau_r + \upsilon_r| + 1} + 3^{p-1},$$

for each $\chi = \{\chi_r\}, \tau = \{\tau_r\}$, and $\upsilon = \{\upsilon_r\} \in \Upsilon$, is an extended b_2-metric on Υ. (see[6], Example2.1).

Definition 2.4.[5] In an extended b_2-metric space (Υ, ρ_φ), the sequence $\{\chi_r\}_{r\in\mathbb{N}}$ in Υ is identified as

1. Convergent if and only if $\exists\, \chi \in \Upsilon$, such that $\lim_{r\to\infty} \rho_\varphi(\chi_r, \chi, \upsilon) = 0$, for every $\upsilon \in \Upsilon$.
   The limit in this instance will be expressed as, $\lim_{r\to\infty} \chi_r = \chi$,
2. Cauchy if and only if
   $$\lim_{r,q\to\infty} \rho_\varphi(\chi_r, \chi_q, \upsilon) = 0, \text{ as, for all } \upsilon \in \Upsilon.$$

Note that: An extended b_2-metric space is regarded as complete if any Cauchy sequence in Υ is a convergent.

Definition 2.5 [7] Let (Υ,ρ_φ) be an extended b_2-metric space, and let $\{\chi_r\}$, $\{\tau_r\}$ sequences in Υ such that for every $\chi, \tau, \upsilon \in \Upsilon$,

$$\lim_{r\to\infty} \rho_\varphi(\chi_r, \tau_r, \upsilon) = \rho_\varphi(\chi, \tau, \upsilon),$$

whenever,

$$\lim_{r\to\infty} \rho_\varphi(\chi_r, \chi, \upsilon) = 0, \text{ and } \lim_{r\to\infty} \rho_\varphi(\tau_r, \tau, \upsilon) = 0.$$

Accordingly, the extended $b_2 -$ metric ρ_φ is defined as continuous on Υ^3.

Remark2.2 An extended b_2-metric ρ_φ is generally not a continuous function of its variables(see [7] Example 2.3, for instance).

Proposition 2.1[5] In an extended b_2 -metric space (Υ, ρ_φ), if ρ_φ a continuous function, then the following assertions hold:

i) The limit of converges sequence is unique,

ii) any subsequence of a sequence $\{\chi_r\}$ that converges to χ in Υ, converges to χ as well.

For more details related to the extended b_2 -metric spaces, one may refer to [5-7].

Definition 2.6. [9,12] For two extended b_2- metric spaces (Υ_1, ρ_1) and (Υ_2, ρ_2). The mapping $\mathcal{L} : \Upsilon_1 \to \Upsilon_2$ is renowned be continuous at $\chi \in \Upsilon_1$, if for any sequence $\{\chi_r\}_{r\in\mathbb{N}}$ in Υ_1 convergent to χ, then $\{\mathcal{L}(\chi_r)\}_{r\in\mathbb{N}}$ convergent to $\mathcal{L}(\chi)$, which means that a mapping $\mathcal{L}$ is continuous at χ if and only if it is sequentially continuous at χ.

3. Main Results

This section illustrates the main findings within the framework of extended $b2$-metric space.

As ρ_φ is not always continuous, so across this section, we will assume that ρ_φ is a continued function.

Theorem 3. 1 Let (Υ, ρ_φ) be complete extended $b2$-metric space. The mapping $\mathcal{L} : \Upsilon \to \Upsilon$, has a unique fixed point in Υ, if the following achieve

i. $\mathcal{L}$ is continuous,

ii.
$$\rho_\varphi{}^2(\mathcal{L}(\chi), \mathcal{L}(\tau), \upsilon) \le \mu_1 \rho_\varphi(\chi, \mathcal{L}(\chi), \upsilon)\rho_\varphi(\tau, \mathcal{L}(\tau), \upsilon) + \mu_2 \rho_\varphi(\chi, \mathcal{L}(\chi), \upsilon)\rho_\varphi(\tau, \mathcal{L}(\chi), \upsilon) + \mu_3\, \rho_\varphi(\tau, \mathcal{L}(\tau), \upsilon) d_\theta(\tau, \mathcal{L}(\chi), \upsilon) + \mu_4 \rho_\varphi(\chi, \mathcal{L}(\tau), \upsilon)\rho_\varphi(\tau, \mathcal{L}(\chi), \upsilon), \quad (3.1)$$

iii. $\lim_{r,s\to\infty} \mu_1\, \varphi(\chi_r, \chi_s, \upsilon) < 1$.

Where, $\mu_1, \mu_2, \mu_3, \mu_4 \ge 0$ with max $\{\mu_1, \mu_4\} < 1$, for all $\chi, \tau, \upsilon \in \Upsilon$.

Proof

Presume χ_0 any element in Υ, and specify the iterative sequence $\{\chi_r\}$, as

$$\chi_r = \mathcal{L}(\chi_{r-1}) = \mathcal{L}^r(\chi_0), \; r = 1,2, \dots .$$

Presuming $\chi_r \ne \chi_{r+1}$, $r = 0,1,2, \dots$, we obtain,

$$\begin{aligned}\rho_\varphi{}^2(\chi_r, \chi_{r+1}, \upsilon) &= \rho_\varphi{}^2(\mathcal{L}(\chi_{r-1}), \mathcal{L}(\chi_r), \upsilon),\\ &\le \mu_1 \rho_\varphi(\chi_{r-1}, \mathcal{L}(\chi_{r-1}), \upsilon)\rho_\varphi(\chi_r, \mathcal{L}(\chi_r), a)\\ &\quad + \mu_2 \rho_\varphi(\chi_{r-1}, \mathcal{L}(\chi_{r-1}), \upsilon)\rho_\varphi(\chi_r, \mathcal{L}(\chi_{r-1}), \upsilon)\\ &\quad + \mu_3 \rho_\varphi(\chi_r, \mathcal{L}(\chi_r), \upsilon)\rho_\varphi(\chi_r, \mathcal{L}(\chi_{r-1}), a)\\ &\quad + \mu_4 \rho_\varphi(\chi_{r-1}, \mathcal{L}(\chi_r), \upsilon)\rho_\varphi(\chi_r, \mathcal{L}(\chi_{r-1}), \upsilon),\\ &= \mu_1 \rho_\varphi(\chi_{r-1}, \chi_r, \upsilon)\rho_\varphi(\chi_r, \chi_{r+1}, \upsilon),\end{aligned}$$

Consequently,

$$\begin{aligned}\rho_\varphi(\chi_r, \chi_{r+1}, \upsilon) &\le \mu_1 \rho_\varphi(\chi_{r-1}, \chi_r, \upsilon),\\ &\le \mu_1{}^2 \rho_\varphi(\chi_{r-2}, \chi_{r-1}, \upsilon),\end{aligned}$$

Continuing, we achieve

$$\rho_\varphi(\chi_r, \chi_{r+1}, \upsilon) \le \mu_1{}^r \rho_\varphi(\chi_0, \chi_1, \upsilon), \qquad \text{for every } \upsilon \in \Upsilon.$$

For $q > r$, applying inquality(4) of definition2.3, we obtain

$$\begin{aligned}\rho_\varphi(\chi_r, \chi_q, \upsilon) &\le \varphi(\chi_r, \chi_q, \upsilon)[\rho_\varphi(\chi_r, \chi_q, \chi_{r+1}) + \rho_\varphi(\chi_q, \upsilon, \chi_{r+1}) + \rho_\varphi(\upsilon, \chi_r, \chi_{r+1})],\\ &\le \varphi(\chi_r, \chi_q, \upsilon)[\mu_1{}^r \rho_\varphi(\chi_0, \chi_1, \upsilon) + \mu_1{}^r \rho_\varphi(\chi_0, \chi_1, \chi_q)] + \varphi(\chi_r, \chi_q, \upsilon)\rho_\varphi(\chi_{r+1}, \chi_q, \upsilon),\\ &\le [\varphi(\chi_r, \chi_q, \upsilon)\mu_1{}^r + \varphi(\chi_r, \chi_q, \upsilon)\varphi(\chi_{r+1}, \chi_q, \upsilon)\mu_1{}^{r+1} + \dots + \varphi(\chi_r, \chi_q, \upsilon)\varphi(\chi_{r+1}, \chi_q, \upsilon) \dots \varphi(\chi_{q-2}, \chi_q, \upsilon)\varphi(\chi_{q-1}, \chi_q, \upsilon)\mu_1{}^{q-1}]\\ &\quad \left(\rho_\varphi(\chi_0, \chi_1, \upsilon) + \rho_\varphi(\chi_0, \chi_1, \chi_q)\right),\\ &= \left(\sum_{i=r}^{q-1} \mu_1{}^i \prod_{j=r}^{i} \varphi(\chi_j, \chi_q, \upsilon)\right)(\rho_\varphi(\chi_0, \chi_1, \upsilon) + \rho_\varphi(\chi_0, \chi_1, \chi_q)), \qquad (3.2)\end{aligned}$$

We have, $\rho_\varphi(\chi_0, \chi_1, \chi_q) = 0$ (See [6], Lemma2.1,page 693).

As a consequence, equation(3.2) turns into

$$\rho_\varphi(\chi_r, \chi_q, \upsilon) \le \left(\sum_{i=r}^{q-1} \mu_1{}^i \prod_{j=r}^{i} \varphi(\chi_j, \chi_q, \upsilon)\right)\rho_\varphi(\chi_0, \chi_1, \upsilon). \quad (3.3)$$

Given that, $\lim_{r,q\to\infty} \mu_1\, \varphi(\chi_r, \chi_q, \upsilon) < 1$, utilizing ratio test for each $q \in \mathbb{N}$, the series $\sum_{i=1}^{\infty} \mu_1{}^i \prod_{j=1}^{i} \varphi(\chi_j, \chi_q, \upsilon)$, converges. Let,

$$\mathfrak{I} = \sum_{i=1}^{\infty} \mu_1{}^i \prod_{j=1}^{i} \varphi(\chi_j, \chi_q, \upsilon),$$

with,

$$\mathfrak{I}_r = \sum_{i=1}^{r} \mu_1{}^i \prod_{j=1}^{i} \varphi(\chi_j, \chi_q, \upsilon).$$

For any $r, q \in \mathbb{N}$, $q > r$, utilizing inequality equation (3.3) indicates,

$$\rho_\varphi(\chi_r, \chi_q, \upsilon) \le \rho_\varphi(\chi_0, \chi_1, \upsilon)[\mathfrak{I}_{q-1} - \mathfrak{I}_{r-1}]. \qquad (3.4)$$

We infer that $\{\chi_r\}$ is a Cauchy sequence in Υ by setting $r \to \infty$ in equation (3.4).

Given that Υ is complete, so $\exists\, \chi \in \Upsilon$, such that $\chi_r \to \chi$.

At the moment, we assert that χ is a fixed point of $\mathcal{L}$.

Given that, $\chi_r \to \chi$, utilizing the continuity of $\mathcal{L}$, and popestion 2.1(ii), we obtain

$$\rho_\varphi(\mathcal{L}(\chi), \chi, \upsilon) = \lim_{r=\to\infty} \rho_\varphi(\mathcal{L}(\chi_r), \chi_r, \upsilon),$$

$$= \lim_{r \to \infty} \rho_{\varphi}(\chi_{r+1}, \chi_r, \upsilon) = \rho_{\varphi}(\chi, \chi, \upsilon) = 0.$$

Therefore, $\rho_{\varphi}(\mathcal{L}(\chi), \chi, \upsilon) = 0$ for every $\upsilon \in \Upsilon$.

So, $\mathcal{L}(\chi) = \chi$. Consequently, χ is a fixed point of $\mathcal{L}$.

To demonstrate the uniqueness: utilizing (iii), again for different points, χ and τ, with $\mathcal{L}(\chi) = \chi$ and $\mathcal{L}(\tau) = \tau$, we get,

$$0 \neq \rho_{\varphi}^{2}(\chi, \tau, \upsilon) = \rho_{\varphi}^{2}(\mathcal{L}(\chi), \mathcal{L}(\tau), \upsilon),$$
$$\leq \mu_4 \rho_{\varphi}(\chi, \mathcal{L}(\tau), \upsilon)\rho_{\varphi}(\tau, \mathcal{L}(\chi), \upsilon),$$
$$= \mu_4 \rho_{\varphi}^{2}(\chi, \tau, \upsilon).$$

Consequently, $(1 - \mu_4)\rho_{\varphi}^{2}(\chi, \tau, \upsilon) \leq 0$, gives $\mu_4 \geq 1$.

That is an inconsistency to $\mu_4 < 1$. So, $\chi = \tau$.
That demonstrates χ is unique.

Theorem 3.2. Let $(\Upsilon, \rho_{\varphi})$ be complete extended $b2$-metric space. The mappings $\mathcal{L}, S : \Upsilon \to \Upsilon$ have a unique common fixed point in Υ, if the following achieve

i) $\mathcal{L}, S$ are continuous,

$ii)$ $\mathcal{L}S = S\mathcal{L}$, with $\mathcal{L}(\chi) \subset S(\chi)$,

$iii)$
$$\rho_{\varphi}^{2}(\mathcal{L}(\chi), \mathcal{L}(\tau), \upsilon) \leq \mu_1 \rho_{\varphi}(S(\chi), \mathcal{L}(\chi), \upsilon)\rho_{\varphi}(S(\tau), \mathcal{L}(\tau), \upsilon) + \mu_2 \rho_{\varphi}(S(\chi), \mathcal{L}(\chi), \upsilon)\rho_{\varphi}(S(\tau), \mathcal{L}(\chi), \upsilon) + \mu_3 \rho_{\varphi}(S(\tau), \mathcal{L}(\tau), \upsilon)\rho_{\varphi}(S(\tau), \mathcal{L}(\chi), \upsilon) + \mu_4 \rho_{\varphi}(S(\chi), \mathcal{L}(\tau), \upsilon)\rho_{\varphi}(S(\tau), \mathcal{L}(\chi), \upsilon), \quad (3.5)$$

iv) $\lim_{r,q \to \infty} \mu_1 \varphi(\chi_r, \chi_q, \upsilon) < 1$,

where $\mu_1, \mu_2, \mu_3, \mu_4 \geq 0$ with $\max\{\mu_1, \mu_4\} < 1$ for all $\chi, \tau, \upsilon \in \Upsilon$.

Proof. Presume χ_0 any element in Υ, given $\mathcal{L}(\chi) \subset S(\chi)$, we can opt $\chi_1 \in \Upsilon$ so that,

$$\mathcal{L}(\chi_r) = S(\chi_{r+1}), \quad r = 0,1,2, \dots .$$

Since,

$$\rho_{\varphi}(S(\chi_{r+1}), S(\chi_{r+2}), \upsilon) = \rho_{\varphi}(\mathcal{L}(\chi_r), \mathcal{L}(\chi_{r+1}), \upsilon) \quad \text{for all } \upsilon \in \Upsilon.$$

Therefore,

$$\rho_{\varphi}^{2}(S(\chi_{r+1}), S(\chi_{r+2}), \upsilon) = \rho_{\varphi}^{2}(\mathcal{L}(\chi_r), \mathcal{L}(\chi_{r+1}), \upsilon).$$
$$\leq \mu_1 \rho_{\varphi}(S(\chi_r), \mathcal{L}(\chi_r), \upsilon)\rho_{\varphi}(S(\chi_{r+1}), \mathcal{L}(\chi_{r+1}), \upsilon)$$
$$+\mu_2 \rho_{\varphi}(S(\chi_r), \mathcal{L}(\chi_r), \upsilon)\rho_{\varphi}(S(\chi_{r+1}), \mathcal{L}(\chi_r), \upsilon)$$
$$+\mu_3 \rho_{\varphi}(S(\chi_{r+1}), \mathcal{L}(\chi_{r+1}), \upsilon)\rho_{\varphi}(S(\chi_{r+1}), \mathcal{L}(\chi_r), \upsilon)$$
$$+ \mu_4 \rho_{\varphi}(S(\chi_r), \mathcal{L}(\chi_{r+1}), \upsilon)\rho_{\varphi}(S(\chi_{r+1}), \mathcal{L}(\chi_r), \upsilon),$$
$$= \mu_1 \rho_{\varphi}(S(\chi_r), S(\chi_{r+1}), \upsilon)\rho_{\varphi}(S(\chi_{n+1}), S(\chi_{r+2}), \upsilon).$$

Accordingly,

$$\rho_{\varphi}(S(\chi_{r+1}), S(\chi_{r+2}), \upsilon) \leq \mu_1 \rho_{\varphi}(S(\chi_r), S(\chi_{r+1}), \upsilon),$$
$$\leq \mu_1^{2}\rho_{\varphi}(S(\chi_{r-1}), S(\chi_r), \upsilon).$$

Continuing, we achieve

$$\rho_{\varphi}(S(\chi_{r+1}), S(\chi_{r+2}), \upsilon) \leq \mu_1^{r+1}\rho_{\varphi}(S(\chi_0), S(\chi_1), \upsilon) \text{ for every } \upsilon \in \Upsilon.$$

For any $q, r \in \mathbb{N}$, $q > r$, we obtain that,

$$\rho_{\varphi}\left(S(\chi_r), S(\chi_q), \upsilon\right)$$
$$\leq \varphi(\chi_r, \chi_q, \upsilon)\left[\rho_{\varphi}\left(S(\chi_r), S(\chi_q), S(\chi_{r+1})\right) + \rho_{\varphi}\left(S(\chi_q), S(\chi_{r+1}), \upsilon\right) + \rho_{\varphi}(S(\chi_r), S(\chi_{r+1}), \upsilon)\right],$$
$$\leq \varphi(\chi_r, \chi_q, a)\left[\mu_1^{r}\rho_{\varphi}\left(S(\chi_0), S(\chi_1), S(\chi_q)\right) + \mu_1^{r}\rho_{\varphi}(S(\chi_0), S(\chi_1), \upsilon) + \rho_{\varphi}(S(\chi_{r+1}), S(\chi_q), \upsilon),\right]$$
$$\vdots$$
$$\leq [\varphi(\chi_r, \chi_q, \upsilon)\mu_1^{r} + \varphi(\chi_r, \chi_q, \upsilon)\varphi(\chi_{r+1}, \chi_q, \upsilon)\mu_1^{r+1} + \cdots + \varphi(\chi_r, \chi_q, \upsilon)\varphi(\chi_{r+1}, \chi_q, \upsilon) \dots \varphi(\chi_{q-2}, \chi_q, \upsilon)\varphi(\chi_{q-1}, \chi_q, \upsilon)\mu_1^{q-1}]$$
$$(\rho_{\varphi}(S(\chi_0), S(\chi_1), \upsilon) + \rho_{\varphi}\left(S(\chi_0), S(\chi_1), S(\chi_q)\right)),$$
$$= \left(\sum_{i=r}^{q-1} \mu_1^{i} \prod_{j=r}^{i} \varphi(\chi_j, \chi_q, \upsilon)\right)\rho_{\varphi}(S(\chi_0), S(\chi_1), \upsilon) + \rho_{\varphi}\left(S(\chi_0), S(\chi_1), S(\chi_q)\right). \quad (3.6)$$

We have,

$$\rho_{\varphi}(S\chi_0, S\chi_1, S\chi_q) \leq \varphi(\chi_0, \chi_1, \chi_q)[\rho_{\varphi}(S\chi_0, S\chi_1, S\chi_{q-1}) + \rho_{\varphi}(S\chi_{q-1}, S\chi_q, S\chi_0) + \rho_{\varphi}(S\chi_{q-1}, S\chi_q, S\chi_1)],$$
$$\leq \varphi(\chi_0, \chi_1, \chi_q)[\rho_{\varphi}(S\chi_0, S\chi_1, S\chi_{q-1}) + \mu^{q-1}\rho_{\varphi}(S\chi_0, S\chi_1, S\chi_0) + \mu^{q-2}\rho_{\varphi}(S\chi_1, S\chi_2, S\chi_1)],$$
$$= \varphi(\chi_0, \chi_1, \chi_q)\rho_{\varphi}(S\chi_0, S\chi_1, S\chi_{q-1}).$$
$$\leq \varphi(\chi_0, \chi_1, \chi_q)\varphi(\chi_0, \chi_1, \chi_{q-1})\rho_{\varphi}(S\chi_0, S\chi_1, S\chi_{q-2}),$$
$$\vdots$$
$$\leq \left(\prod_{j=2}^{q} \varphi(\chi_0, \chi_1, \chi_{q-j+2})\right)\rho_{\varphi}(S\chi_0, S\chi_1, S\chi_1),$$
$$= 0.$$

So, $\rho_{\varphi}(S\chi_0, S\chi_1, S\chi_q) = 0$.

As a consequence, equation (3.6) turns into

$$\rho_{\varphi}\left(S(\chi_r), S(\chi_q), \upsilon\right) \leq \left(\sum_{i=r}^{q-1} \mu_1^{i} \prod_{j=r}^{i} \varphi(\chi_j, \chi_q, \upsilon)\right)\rho_{\varphi}(S(\chi_0), S(\chi_1), a). \quad (3.7)$$

In the same manner in equation(3.3), we infer that, $\{S(\chi_r)\}_{r \in \mathbb{N}}$ is a Cauchy sequence in Υ. Given that Υ is complete. So there is $\chi \in \Upsilon$ such that $S(\chi_r) \to \chi$.

Utilizing (ii) of Proposition 2.1, we infer

$$\mathcal{L}(\chi_r) = S(\chi_{2r+1}) \to \chi.$$

Given that $S\mathcal{L} = \mathcal{L}S$, utilizing the continuity of $S, \mathcal{L}$, we obtain

$$\mathcal{L}(\chi) = \mathcal{L}\left(\lim_{r \to \infty} S(\chi_r)\right) = \lim_{r \to \infty} \mathcal{L}(S(\chi_r))$$
$$= \lim_{r \to \infty} S(\mathcal{L}(\chi_r)) = S\left(\lim_{r \to \infty} \mathcal{L}(\chi_r)\right) = S(\chi).$$

Consequently, $\mathcal{L}(\chi) = S(\chi)$.

At the moment, we assert that χ is a fixed point of S.

Presuming that, $S(\chi) \neq \chi$. Therefore $\exists\, \upsilon \in \Upsilon$, so that

$\rho_{\varphi}(S(\chi), \chi, \upsilon) \neq 0$.

Utilizing (iii) we obtain,

$$0 \neq \rho_{\varphi}^{2}(S(\chi), \chi, \upsilon) = \lim_{r \to \infty} \rho_{\varphi}^{2}(\mathcal{L}(\chi), \mathcal{L}(\chi_r), \upsilon).$$
$$\leq \mu_4 \rho_{\varphi}^{2}(S(\chi), \chi, \upsilon).$$

Therefore, $(1 - \mu_4)\rho_{\varphi}^{2}(S(\chi), \chi, \upsilon) \leq 0$, gives $\mu_4 \geq 1$.

That is an inconsistency to $\mu_4 < 1$. Indicates that $S(\chi) = \chi$.

Accordingly, $\mathcal{L}(\chi) = S(\chi) = \chi$.

Consequently, χ is a common fixed point of $\mathcal{L}$ and S.

To demonstrate the uniqueness: utilizing (iii), again for different points, χ and τ, with $\mathcal{L}(\chi) = S(\chi) = \chi$ and $\mathcal{L}(\tau) = S(\tau) = \tau$, indicates that $\chi = \tau$.

That demonstrates χ is unique.

Theorem 3.3 Let (Υ, ρ_φ), be a complete extended b_2-metric space. The mappings $\mathcal{L}, S, F: \Upsilon \to \Upsilon$, have a unique common fixed point in Υ, if the following achieve

i) $\mathcal{L}, S$ and F are continuous mappings,

ii) $\mathcal{L}F = F\mathcal{L}$, $SF = FS$, with $\mathcal{L}(\chi) \subset F(\chi), S(\chi) \subset F(\chi)$,

iii)
$$\rho_\varphi^2(\mathcal{L}(\chi), S(\tau), \upsilon) \le \mu_1 \rho_\varphi(F(\chi), \mathcal{L}(\chi), \upsilon)\rho_\varphi(F(\tau), S(\tau), \upsilon) + \mu_2 \rho_\varphi(F(\chi), \mathcal{L}(\chi), \upsilon)\rho_\varphi(F(\tau), \mathcal{L}(\chi), \upsilon) + \mu_3 \rho_\varphi(F(\tau), S(\tau), \upsilon)\rho_\varphi(F(\tau), \mathcal{L}(\chi), \upsilon) + \mu_4 \rho_\varphi(F(\chi), S(\tau), \upsilon)\rho_\varphi(F(\tau), \mathcal{L}(\chi), \upsilon), \quad (3.9)$$

iv) $\lim_{r,q\to\infty} \mu_1\, \varphi(\chi_r, \chi_q, \upsilon) < 1$,

where $\mu_1, \mu_2, \mu_3, \mu_4 \ge 0$ with $\max\{\mu_1, \mu_4\} < 1$ for all $\chi, \upsilon, \tau \in \Upsilon$.

Proof. Presume χ_0 any element in Υ, by(ii) we can opt $\chi_1, \chi_2 \in \Upsilon$, so that

$F(\chi_{2r+1}) = \mathcal{L}(\chi_{2r})$, and $F(\chi_{2r+2}) = S(\chi_{2r+1})$, $r = 0,1,2,\dots$.

Therefore,
$$\rho_\varphi^2(F(\chi_{2r+1}), F(\chi_{2r+2}), \upsilon) = \rho_\varphi^2(\mathcal{L}(\chi_{2r}), S(\chi_{2r+1}), \upsilon).$$
$$\le \mu_1\, \rho_\varphi(F(\chi_{2r}), F(\chi_{2r+1}), \upsilon)\rho_\varphi(F(\chi_{2r+1}), F(\chi_{2r+2}), \upsilon).$$

Accordingly,
$$\rho_\varphi(F(\chi_{2r+1}), F(\chi_{2r+2}), \upsilon) \le \mu_1 \rho_\varphi(F(\chi_{2r}), F(\chi_{2r+1}), \upsilon)$$
$$\le \mu_1^2 \rho_\varphi(F(\chi_{2r-1}), F(\chi_{2r}), \upsilon).$$

Continuing, we achieve
$$\rho_\varphi(F(\chi_{2r+1}), F(\chi_{2r+2}), \upsilon) \le \mu_1^r \rho_\varphi(F(\chi_0), F(\chi_1), \upsilon), \text{for all } \upsilon \in \Upsilon.$$

For any $q, r \in \mathbb{N}$, $q > r$, we obtain that,
$$\rho_\varphi(F(\chi_r), F(\chi_q), \upsilon) \le \left(\sum_{i=r}^{q-1} \mu_1^i \prod_{j=r}^{i} \varphi(\chi_j, \chi_q, \upsilon)\right)\rho_\varphi(F(\chi_0), F(\chi_1), a). \quad (3.10)$$

In the same manner in equation(3.3), we infer that, $\{F(\chi_r)\}_{r\in\mathbb{N}}$ is a Cauchy sequence in Υ. Given that Υ is complete. So there is $\chi \in \Upsilon$ such that $F(\chi_r) \to \chi$.

Utilizing (ii) of Proposition 2.1, we infer
$$\mathcal{L}(\chi_{2r}) = F(\chi_{2r+1}) \to \chi, \text{ and } S(\chi_{2r+1}) = F(\chi_{2r+2}) \to \chi.$$

Given that $F\mathcal{L} = \mathcal{L}F$, $SF = FS$, and by utilizing the continuity of $S, \mathcal{L}$, and F, we obtain
$$\mathcal{L}(\chi) = \mathcal{L}\left(\lim_{r\to\infty} F(\chi_r)\right) = \lim_{r\to\infty} \mathcal{L}(F(\chi_{2r})),$$
$$= \lim_{r\to\infty} F(\mathcal{L}(\chi_{2r})) = F\left(\lim_{r\to\infty} \mathcal{L}(\chi_{2r})\right) = F(\chi).$$

Consequently, $\mathcal{L}(\chi) = F(\chi)$. In the same approach, we demonstrate that $F(\chi) = S(\chi)$.

And so, $\mathcal{L}(\chi) = S(\chi) = F(\chi)$.

Presuming that, $F(\mathcal{L}(\chi)) \ne \mathcal{L}(\chi)$. Therefore $\exists\, \upsilon \in \Upsilon$, so that $\rho_\varphi\ (F(\mathcal{L}(\chi)), \mathcal{L}(\chi), \upsilon) \ne 0$. Utilizing (iii) we obtain,
$$(1 - \mu_4)\rho_\varphi^2(F(\mathcal{L}(\chi)), \mathcal{L}(\chi), \upsilon) \le 0, \quad \text{gives } \mu_4 \ge 1.$$

That is an inconsistency to $\mu_4 < 1$. Indicates that $F(\mathcal{L}(\chi)) = \mathcal{L}(\chi)$.

Accordingly, $F(\mathcal{L}(\chi)) = \mathcal{L}(\mathcal{L}(\chi)) = S(\mathcal{L}(\chi)) = \mathcal{L}(\chi)$.

Consequently, $\mathcal{L}(\chi)$ is a common fixed point of $\mathcal{L}, S$ and F.

Condition(iii) directly leads to uniqueness.

4. Discussion

In this section, the related findings debated in relation with previous ensues in the literature, and we obtain the following:

i) If $\mu_2, \mu_3 = 0$, and all other conditions remain the same as in Theorem 3.1 (analogously, in Theorem 3.2 and 3.3), then Theorem 3.1 (analogously, in Theorem 3.2 and 3.3) remains viable,

ii) the corresponding findings of Alkaleeli.,et.al.[9] in the frame of b_2 -metric space, by letting $\varphi(\chi, \tau, \upsilon) = s \ge 1$, constant with $\mu_1 s < 1$ in Theorem 3.1 (analogously, in Theorem 3.2),

iii) the corresponding findings of Aage.,et.al.[8] in the frame of 2-metric space, by letting $\varphi(\chi, \tau, \upsilon) = 1$ in Theorem 3.1 (analogously, in Theorem 3.2 and 3.3).

5. Conclusion

The current study aimed to investigate the existence and uniqueness of common fixed points using a novel concept to distance, known as extended b_2-metric space, which Elmabrouk and Alkaleeli [5] most recently established. This concept is more general than the 2-metric and b_2 -metric. The main results of this study are to generalize the corresponding ones in the previous publications achieved by Aage.,et.al. [9]. Numerous researchers in this field may be motivated through this study to investigate other theories in extended b_2-metric spaces.

6.References

[1]- Gähler, V.S. (1963). 2-metrische Räume und ihre topologische Struktur. Math.Nachr , pp. 115-118.

[2]- Czerwik, S. (1993). Contraction mapping in b-metric spaces. Acta Mathematica et Informatica Universitatis Ostravensis , pp. 5-11.

[3]- Kamran, T., Samreen, M., Ain, O.U. (2017). A generalization of b-metric space and some fixed point theorems. Mathematics 5(2),19.

[4]- Mustafa, Z., Parvaech, V., Roshan, J.R. and Kadelburg Z. (2014). b_2-metric spaces and some Fixed Point Theorems.Fixed Point Theory and Applications ,pp.144. https://doi.org/10.1186/1687-I812-2014-144.

[5]- Elmabrok, A. S., & Alkaleeli, S. R. (2018). Generalization of b_2-metric spaces and some fixed point theorems. Libyan Journal of Science&Technology, 51–55.

[6]- Alkaleeli. S.R., Dabnoun. N.M.O.,(2021). On Generalized Some Fixed Point Theorems in Extended b_2–Metric Spaces, International Journal of Multidisciplinary Sciences and Advanced Technology, issue 1(2021) 691–702.

[7]- Elmabrok, A. S., & Alkalee i. S.R.,(2021).Non unique fixed point on extended b_2-metric spaces , Libyan Journal of Science & Technology Volume 13, Issue 2 (2021), pp.85-90.

[8]- Aage CT, Salunke JN.(2008). A Note on Fixed Point and Common Fixed Point Theorems in 2-Metric Spaces. Int. Journal of Math. Analysis; 2(28): 1381-1392.

[9]- Alkaleeli.S.R., Meriki.N. F.M.,& Abdulsalam.S.A.S.,(2019). Unique Fixed Point Theorems In b_2- Complete Metric Spaces, Al-Mukhtar Journal of Educational Sciences, pp1-12.

[10]-Huang, H.; Singh, Y.M.; Khan, M.S.; Radenović, S.(2021) Rational Type Contractions in Extended b-Metric Spaces. Symmetry 2021, 13, 614. https://doi.org/10.3390/sym13040614.

[11]-Hassen Aydi, Muhammad Aslam, Dur-e-Shehwar Sagheer, Samina Batul, Rashid Ali, Eskandar Ameer,(2021). "Kannan-Type Contractions on New Extended -Metric Spaces", Journal of Function Spaces, vol. 2021, Article ID 7613684, 12 pages. https://doi.org/10.1155/2021/7613684.
[12]-Zaid, M. Abd Ghafur, B., Stojan, R., Vildan,. (2015). Some Fixed Points Results of b_2-metric spaces. Far est journal of mathematical sciences(FJMS), pp. 533-548. http://dx.doi.og/10.17654.
[13]-Venkata ,S. (2001). Some Fixed Point Theorem in Metric and 2-Metric Spaces. IJMMS , pp. 625-636. ijmms.hindawu.com.
[14]-Naidu, S. (2001). Some fixed point theorems in metric and 2-metric spaces. International Journal of Mathematics and Mathematical Sciences, 28(11), 625-636.