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The stability of eigenvalues and eigenvectors and their impact on differential systems

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A B S T R A C T

In this article, we apply the stability of eigenvalues and eigenvectors and their impact on differential systems. To achieve this goal, the eigenvalues and eigenvectors are studied and their differential systems, nature in terms of being different real values, compound eigenvalues, or equal eigenvalues.And to identify how to solve linear differential systems with fixed coefficients with the initial condition a complete solution, which depends on the eigenvalues and the corresponding eigenvectors, finding the general solution and the geometry of teigenvectors graphically and the effect of theigenvalues for the three eigenvalues cases, by drawing the paths and the phase plane and clarifying the state of equilibrium contract and stability

استقرار القيم الذاتية والمتجهات الذاتية و اثرها في المنظومات التفاضلية

*مبروكة الشارف غيث و عواطف هدية منصور أبوقفة

قسم الرياضيات ، كلية العلوم والموارد الطبيعية ، جامعة الجفارة ، ليبيا

الكلمات المفتاحية:
لقيم الذاتية
المتجهات الذاتية
الأستقرارية
المنظومات التفاضلية

الملخص

يهدف هذا البحث إلى معرفة استقرار القيم الذاتية والمتجهات الذاتية وأثرها في المنظومات التفاضلية، ولتحقيق هذا الهدف يتم دراسة القيم والمتجهات الذاتية وطبيعتها من حيث كونها قيم حقيقية مختلفة أو قيم ذاتية مركبة أو قيم ذاتية متساوية. والتعرف على كيفية حل المنظومات التفاضلية والمتجهات الذاتية المقابلة لها وإيجاد الحل العام وهندسة المتجهات الذاتية بيانياً وتأثير القيم الذاتية لحالات القيم الذاتية الثلاثة، وذلك من خلال رسم المسارات ومستوى الطور وتوضيح حالة التوازن والهقد والاستقر.

1- Introduction

The eigenvalues are reals values that could be found through square matrixes as well as from the system of differential equations with fixed coefficients, and by using the system solution, we find the eigenvectors that we used in the graphical geometry of the differential system.

The study of differential equations that depends oneigenvalues and eigenvectors through solutions of equations and differential systems when the independent variable approaches infinity, and on this basis differential equations are classified, and using this classification can identify the nature of the solution without resorting to calculating that solution, by drawing paths And clarify the state of balance, contract and stability [1,4].

The study of ordinary differential equations begins with linear equations with constant coefficients for two reasons:

First: Studying this type of equations by dispensing with the issue of the existence and starting point of defining the solution and initiating an engineering study of solutions to differential equations and developing the number of this study later.

The second: Studying the geometric properties (topically) of non-linear equations under general conditions by replacing the non-linear equation with its linear part only and neglecting the non-linear part [2].

2- How to calculate eigenvalues and eigenvectors

To get the eigenvalues of a $\{\lambda_i\}_{i=1}^{n}$ matrix $A \in M_n(\mathcal{R})$ where $A = [aij]_{i=1,j=1}^{n}$. The determinant is decoded to calculate its value $|A - \lambda I_n| = 0$ any specified;

$$\begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22-\lambda} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ a_{31} & a_{32} & \cdots & a_{nn}-\lambda \end{vmatrix} = 0$$

So, we get a polynomial equation of degree n in λ. By solving this equation, we get a number n of eigenvalues $\{\lambda_i\}_{i=1}^{n}$,and to get the eigenvectors, we find that for every eigenvalue λ_i there is an eigenvector C_i so that $AC_i = \lambda_i C_i$ for each λ_i ,we have to solve the system of homogeneous linear algebraic equations [3].

$$(A - \lambda_i I_n)C_i = 0, \quad \forall i = 1,2,3,\dots,n, \qquad (2.1)$$

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Since the matrix is $(A - \lambda_i I_n)$ anomalous, the solution of this system is not a single $C_i$eigenvector, it is a blank of eigenvectors, for every eigenvalue λ_i there is an infinite number of eigenvectors C_i, there are three cases of eigenvalues λ:

Case 1: different real eigenvalues

If $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n$, and by substituting into the characteristic Eq. (2.1) we get eigenvectors $C_1, C_2, \ldots, C_n$.

Case 2: composite eigenvalues

If λ_1, λ_2 two complex numbers are conjugated and by $\lambda_1 = a + ib$, $\lambda_2 = a - ib$, substituting into the characteristic equation 2.1 we get eigenvectors in the form of complex numbers $C = C_1 + iC_2$.

Case 3: duplicate real eigenvalues

If $\lambda_1 = \lambda_2 = \cdots = \lambda_n$and by substituting into the characteristic Eq. (2.1) we get eigenvectors $C_1, C_2, \ldots, C_n$. [4-5]

Result 1: If the eigenvalues of matrix A have different scalar values, then the corresponding eigenvectors for each eigenvalue are linearly independent.

Result 2: If the eigenvalues of matrix A are equal, then the corresponding eigenvectors for each eigenvalue may be linearly independent or not linearly independent.

3- Systems of linear differential equations with constant coefficients.

The system of linear differential equations of the first order with constant coefficients has the following form:

$$\begin{cases} \frac{d}{dt}x_1(t) = a_{11}x_1(t) + a_{12}x_2(t) + \cdots + a_{1n}x_n(t) \\ \frac{d}{dt}x_2(t) = a_{21}x_1(t) + a_{22}x_2(t) + \cdots + a_{23}x_n(t) \\ \quad \vdots \qquad\qquad\qquad \vdots \\ \frac{d}{dt}x_3(t) = a_{31}x_1(t) + a_{32}x_2(t) + \cdots + a_{3n}x_n(t) \end{cases}, \quad (3.1)$$

where t independent variable while $x_1(t), x_2(t), \ldots, x_3(t)$ dependent variables and coefficients $a_{ij} \in \mathscr{R}$ where $i, j = 1,2, \ldots, n$ is called $|A - \lambda I| = 0$ a polynomial equation in λ degree n, and therefore it has n eigenvalues of the matrix A, from which we get solution of the system after setting the values of the eigenvectors C corresponding to the eigenvalues λ [6].

3-1 Phase level

The solutions of the differential equation;

$$X' = AX, \qquad (3.1.1)$$

Are curves in space $\mathcal{R}^n$,, so that each cure is a function and not just a set of points, the solution to the equation 3.2 which is X is the movement of a point through time.

So that the position of the point at moment t is $X(t)$.

The differential Eq. (3.1.1) defines a law for the motion of points. When the position of the point is X it will move with velocity AX, and thus in space it $\mathcal{R}^n$ is called the phase plane of the differential equation $X' = AX$. Therefore, every point X of the phase plane has a vector, which is the velocity vector AX, [9].

$$X' \to AX = F(x), \qquad (3.1.2)$$

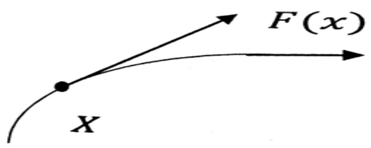

Fig. 1: vector field

3-2 Point path

If x_0 a point in $\mathcal{R}^n$, the set of points tracing the solution to the differential equation $X' = AX$ that passes through the point x_0 when it is $t = 0$ called the path of the point x_0, that is the path of point $A(t) = x_0$, where $t \in \mathcal{R}, X(0) = x_0$. The set of all differential equation paths is called phase images [8].

3-3 Equilibrium point

To find the equilibrium solution for the system of homogeneous differential equations and $X' = AX$ we not that $X = 0$ it is a solution to the system of homogeneous differential equilibrium and this solution is called the equilibrium solutions of the system, and that the equilibrium solutions are those solutions in which there are and $AX = 0$ let us suppose that A is an anomalous matrix, and therefore it will have one solution which is $X = 0$, and also we will have only one equilibrium solution [10].

For the next order;

$$x_1' = a_{11}x_1 + a_{12}x_2 \,,$$
$$x_2' = a_{21}x_1 + a_{22}x_2 \,,$$
$$X' = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} X \text{ where } A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \qquad (3.3.1)$$

The system solutions are as follows:

$$X = \begin{bmatrix} X_1(t) \\ X_2(t) \end{bmatrix},$$

It is our only equilibrium solution $X = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

4- Solutions the eigenvalues of the matrix A

4.1 On Solving Eq. (3.3.1) using the different eigenvalues

We get solutions;

$$X_1(t) = C_1 e^{\lambda_1 t} \;,\; X_2(t) = C_2 e^{\lambda_2 t} \;, \ldots,\; X_n(t) = C_n e^{\lambda_n t}, \; (4.1.1)$$

And the general solution is;

$$X(t) = \alpha_1 C_1 e^{\lambda_1 t} + \alpha_2 C_2 e^{\lambda_2 t} + \cdots + \alpha_n C_n e^{\lambda_n t}, \; (4.1.2)$$

4.2 On solving Eq. (3.3.1) using the complex eigenvalues

We get the following solution;

$$X(t) = (C_1 + iC_2)e^{(a+ib)t}, \qquad (4.2.3)$$

Applying Euler's, which states $\Upsilon^{i\theta} = cos\theta + isin\theta$:

$$X(t) = e^{at}\left[\left(C_1 cos(bt) - C_2 sin(bt)\right) - i\left(C_2 cos(bt) + C_1 sin(bt)\right)\right],$$

(4.2.4)

4.3 Equal real values

The general solution λ is for the first solution to be;

$$X_1(t) = Ce^{\lambda t}, \qquad (4.3.1)$$

The second solution is;

$$X_2(t) = te^{\lambda t}C + e^{\lambda t}\rho, \qquad (4.3.2)$$

It is an unknown vector ρ that we will need to determine [7],

$$X(t) = \alpha_1 e^{\lambda t}C + \alpha_2\left[te^{\lambda t}c + e^{\lambda t}\rho\right], \qquad (4.3.3)$$

5- Examples and results

Example 5-1

Find the solution to the following system with the initial condition:

$$x_1'(t) = x_1(t) + 2x_2(t) \quad , \quad x_1(0) = 0,$$
$$x_2'(t) = 3x_1(t) + 2x_2(t) \quad , \quad x_2(0) = -4,$$

Solution;

The system can be placed on a matrix image $X' = AX$;

$$X' = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} X, where\ A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}, is\ X(0) = \begin{bmatrix} 0 \\ -4 \end{bmatrix},$$

Characteristic equation for matrix A is $|A - \lambda I_2| = 0$.

The eigenvalues of two different real numbers $\lambda_1 = -1$, $\lambda_2 = 4$. To get the eigenvector versus the eigenvalue $\lambda_1 = -1$. we find the solution to the system $(A - \lambda_1 I_2)C_1 = 0$.

In this case, the eigenvector is $C_1 = \begin{bmatrix} -C_2 \\ C_2 \end{bmatrix}$. So, we choose $C_2 = 1$ it $C_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

Here is the first solution: $X_1(t) = C_1 e^{\lambda_1 t} \Rightarrow X_1(t) = \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-t}$.

To get the eigenvector versus the eigenvalue $\lambda_2 = 2$, we find the solution to the system $(A - \lambda_2 I_2)C_2 = 0$.

In this case, the eigenvector is $C_2 = \begin{bmatrix} \frac{2}{3}C_2 \\ C_2 \end{bmatrix}$, then we choose $C_2 = 3$ it $C_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$,, and the second solution is $X_2(t) = C_2 e^{\lambda_2 t} \Rightarrow X_2(t) = \begin{bmatrix} 2 \\ 3 \end{bmatrix} e^{4t}$.

So, the general solution;

$$X(t) = \alpha_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-t} + \alpha_2 \begin{bmatrix} 2 \\ 3 \end{bmatrix} e^{4t},$$

Now, to find the constants by applying the initial condition;

$$X(0) = \begin{bmatrix} 0 \\ -4 \end{bmatrix} = \alpha_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-(0)} + \alpha_2 \begin{bmatrix} 2 \\ 3 \end{bmatrix} e^{4(0)},$$

Such as;

$-\alpha_1 + 2\alpha_2 = 0,$

$\alpha_1 + 3\alpha_2 = -4,$

Then we are gets $\alpha_1 = \frac{-8}{5}$, $\alpha_2 = \frac{-4}{5}$. The solution to a system is;

$$X_1(t) = \frac{8}{5}e^{-t} - \frac{8}{5}e^{4t},$$
$$X_2(t) = \frac{-8}{5}e^{-t} - \frac{8}{5}e^{4t},$$

When draw a picture of a system;

$$X' = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} X,$$

We get;

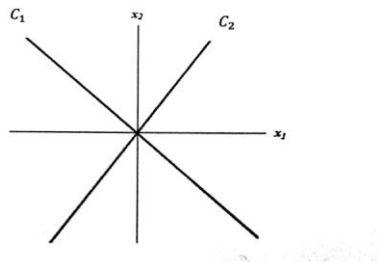

Fig. 2

If it $\alpha_2 = 0$, then the solution is a vector multiplied by an exponential function and the exponent affects the size of the vector and the constant will affect both the sign and the size of the vector, as t increases the path will move towards the origin.

If $\alpha_1 = 0$ it going to have a path parallel to the vector C_2 and since the exponent increases as t increases, then the path moves away from the origin.

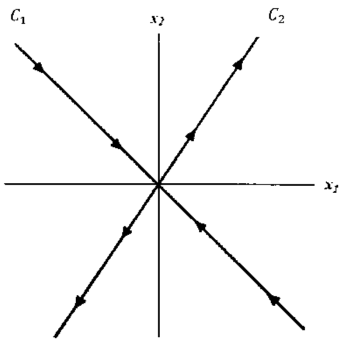

Fig. 3

If $\alpha_1 \neq 0$, $\alpha_2 \neq 0$, then we have two cases.

If t is negative and large, the portion containing negative eigenvalues will dominate the solution, moving the path in the same direction. If t is positive and the eigenvalue portion of the solutions is large, then the path moves in the same direction

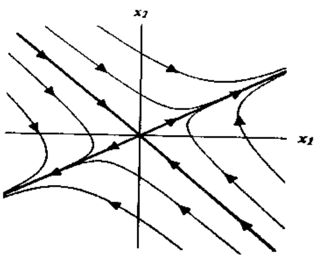

Fig. 4

The equilibrium solution is called the (0,0), saddle point and unstable in this case the instability of solutions moves away from the origin as t increases.

Example 5-2

Find the solution to the following system with initial condition;

$$x_1'(t) = -5x_1(t) + x_2(t) \quad , \quad x_1(0) = 1,$$
$$x_2'(t) = 4x_1(t) - 2x_2(t) \quad , \quad x_2(0) = 2,$$

The system can be placed on a matrix image $X' = AX$;

$$X' = \begin{bmatrix} -5 & 1 \\ 4 & -2 \end{bmatrix} X, where\ A = \begin{bmatrix} -5 & 1 \\ 4 & -2 \end{bmatrix}, is\ X(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix},$$

Characteristic equation for matrix A is $|A - \lambda I_2| = 0$. The eigenvalues of two different real number $\lambda_1 = -1$, $\lambda_2 = -6$.

To get the eigenvector versus the eigenvalue $\lambda_1 = -1$, we find the solution to the system $(A - \lambda_1 I_2)C_1 = 0$. In this case, the eigenvector is $C_1 = \begin{bmatrix} C_1 \\ 4C_1 \end{bmatrix}$,, so, we choose $C_1 = 1$ it $C_1 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$. Here is the first solution $X_1(t) = C_1 e^{\lambda_1 t} \Rightarrow X_1(t) = \begin{bmatrix} 1 \\ 4 \end{bmatrix} e^{-t}$.

And when we but $\lambda_2 = -6$, we get the solution of the system $(A - \lambda_2 I_2)C_2 = 0$. In this case, the eigenvector is $C_2 = \begin{bmatrix} -C_2 \\ C_2 \end{bmatrix}$, so, we choose $C_2 = 1$ it $C_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. Then, the second solution is $X_2(t) = C_2 e^{\lambda_2 t} \Rightarrow X_2(t) = \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{4t}$.

So, the general solution;

$$X(t) = \alpha_1 \begin{bmatrix} 1 \\ 4 \end{bmatrix} e^{-t} + \alpha_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-6t},$$

Now, to find the constants by applying the initial condition;

$$X(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \alpha_1 \begin{bmatrix} 1 \\ 4 \end{bmatrix} e^{-(0)} + \alpha_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-6(0)},$$

Such as;

$\alpha_1 - \alpha_2 = 1,$

$$4\alpha_1 + \alpha_2 = 2,$$

We get the points $\alpha_1 = \frac{3}{5}$, $\alpha_2 = \frac{-2}{5}$, then the solutions are;

$$X_1(t) = \frac{3}{5} e^{-t} - \frac{2}{5} e^{-6t},$$
$$X_2(t) = \frac{12}{5} e^{-t} - \frac{2}{5} e^{-6t},$$

Then drawn a picture of the system;

$$X' = \begin{bmatrix} -5 & 1 \\ 4 & -2 \end{bmatrix} X,$$

Both eigenvalues are negative, so, all paths move toward the origin as t increases.

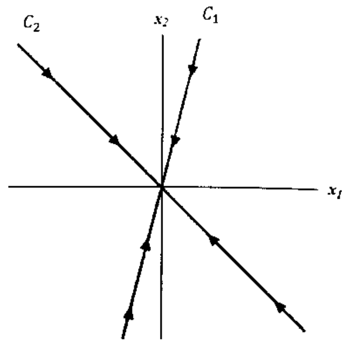

Fig. 5

We note that the second eigenvalue is greater than the first eigenvalue for all large and the second eigenvalue of positive value of t is smaller than the first eigenvalue solution.

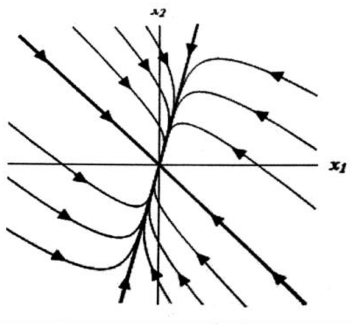

Fig. 6

The equilibrium solution is called (0,0) asymptotically stable, and the equilibrium solution is asymptotically stable if all paths move towards the origin as t increase.

Note: the k cannot be unstable in example 5-2, if both eigenvalues are positive, then all paths will move away from the origin, in which case the equilibrium is unstable.

Example 5-3

Find the solution to the following system with the initial condition:

$$x_1' = 3x_1 + 9x_2 \quad , \quad x_1(0) = 2,$$
$$x_2' = -4x_1 - 3x_2 \quad , \quad x_2(0) = -4,$$

Solution

The system can be placed on a matrix image $X' = AX$

$$X' = \begin{bmatrix} 3 & 9 \\ -4 & -3 \end{bmatrix} X, \quad where\ A = \begin{bmatrix} 3 & 9 \\ -4 & -3 \end{bmatrix},$$
$$is\ \ X(0) = \begin{bmatrix} 2 \\ -4 \end{bmatrix},$$

Characteristic equation for matrix A is $|A - \lambda I_2| = 0$. The eigenvalues of two different real numbers $\lambda_1 = 3\sqrt{3i}$, $\lambda_2 = -3\sqrt{3i}$, when $\lambda_1 = 3\sqrt{3i}$ we find the solution to the system $(A - \lambda_1 I_2)C_1 = 0$.

So, the first eigenvector is $C_1 = \begin{bmatrix} C_1 \\ \frac{-1}{3}(1 - \sqrt{3i})C_1 \end{bmatrix}$, we choose $C_1 = 3$ it $C_1 = \begin{bmatrix} 3 \\ -1 + \sqrt{3i} \end{bmatrix}$.

Her is the first solution $X_1(t) = C_1 e^{\lambda_1 t} \Rightarrow X_1(t) = \begin{bmatrix} 3 \\ -1 + \sqrt{3i} \end{bmatrix} e^{(3\sqrt{3i})t}$. So, the eigenvector is $C_2 = \begin{bmatrix} 3 \\ -1 - \sqrt{3i} \end{bmatrix}$. Then the second solution is;

$$X_2(t) = C_2 e^{\lambda_2 t} \Rightarrow X_2(t) = \begin{bmatrix} 3 \\ -1 - \sqrt{3i} \end{bmatrix} e^{(-3\sqrt{3i})t},$$

Using Euler's formula to get the complex number from the exponential from;

$$X_1(t) = \left[cos(3\sqrt{3t}) + isin(3\sqrt{3t})\right] \begin{bmatrix} 3 \\ -1 - \sqrt{3i} \end{bmatrix},$$

So, the general solution to the system has complex limits;

$$X(t) = \alpha_1 \vec{u}(t) + \alpha_2 \vec{v}(t),$$

Now, we need to find the constants by applying initial condition

$$X(0) = \begin{bmatrix} 2 \\ -4 \end{bmatrix} = \alpha_1 \begin{bmatrix} 3 \\ -1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ \sqrt{3} \end{bmatrix},$$

Such as;

$3\alpha_1 = 2,$

$-\alpha_1 + \sqrt{3}\alpha_2 = 4,$

We get the points $\alpha_1 = \frac{2}{3}$, $\alpha_2 = \frac{-10}{3\sqrt{3}}$, then the solution of a system is;

$$X_1(t) = \frac{2}{3} \begin{bmatrix} 3cos(3\sqrt{3}t) \\ -cos(3\sqrt{3}t) - \sqrt{3}sin(3\sqrt{3}t) \end{bmatrix} - \frac{10}{3\sqrt{3}} \begin{bmatrix} 3sin(3\sqrt{3}t) \\ -sin(3\sqrt{3}t) + \sqrt{3}cos(3\sqrt{3}t) \end{bmatrix},$$

We draw a picture of the system;

$$X' = \begin{bmatrix} 3 & 9 \\ -4 & -3 \end{bmatrix} X,$$

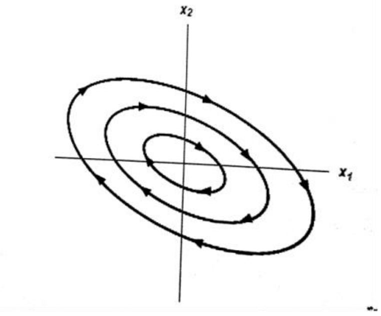

Fig. 7

The solutions of paths will be circles or ellipses centered at the origin and the only thing we need is whether they rotate clockwise or anticlockwise. We choose a value by $X(t)$ substituting in a system, we get a vector that is tangent to the path and indicates the direction in which the path is be $X = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and in compensation be $X' = \begin{bmatrix} 3 & 9 \\ -4 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ -4 \end{bmatrix}$.

At the point (1,0) in the phase plane the path it be point in a descending direction, and this happens if the paths are going in a clockwise direction.

The solution to the equilibrium in this case is called the stationary center.

Conclusion

In this paper, we have applied the stability of eigenvalues and eigenvectors and their impact on differential systems. We conclude that results with some figures expressing the behavior of the obtained solutions by examples which give some perspective readers how the behavior solutions are produced.

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