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Absolutely Continuous Invariant Measures for Piecewise Expanding Chaotic Transformations in $\mathbb{R}^n$ with Summable Oscillations of Derivative

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ABSTRACT

This paper presents a detailed investigation of absolutely continuous invariant measures (ACIMs) for piecewise expanding chaotic transformations in $\mathbb{R}^n$, with particular attention paid to the case where the derivative has summable oscillations. ACIMs are important objects in the study of dynamical systems, as they provide a way to understand the long-term behavior of trajectories and the statistical properties of the system. The paper covers a range of important topics related to ACIMs, including the boundedness condition, distortion condition, localization condition, and Schmitt's condition. It also discusses the Perron-Frobenius operator, which plays a critical role in the existence and properties of ACIMs. The main result of the paper is the proof that the Perron-Frobenius operator is constrictive, which implies the existence of a finite number of ergodic ACIMs that satisfy Schmitt's condition and a condition dependent on the defining partition. This finding has significant implications for the understanding of complex systems and the advancement of research in this field. The paper also discusses the relationship between ACIMs and dynamical systems, highlighting the role of ACIMs in ergodic theory. Overall, this paper provides a valuable reference for researchers interested in the study of ACIMs and their significance in the analysis of dynamical systems and ergodic theory.

مقاييس ثابتة مستمرة تماما لتوسيع التحولات الفوضوية متعددة في $\mathbb{R}^n$ مع تذبذبات قابلة للمجموع للمشتقة

عبدالسلام عصمان بيت المال

قسم الرياضيات، كلية العلوم، جامعة سبها، ليبيا

الكلمات المفتاحية
آسيم
شرط الحدود
حالة التشويه
حالة التعريب
حالة شميت
مقاييس ثابتة
مشغل بيرون فروبينيوس

الملخص

تقدم هذه الورقة تحقيقًا مفصلًا للمقاييس الدائمة المطلقة المتصلة (ACIMs) للتحويلات الفوضوية المتسعة القطعية في $\mathbb{R}^n$ ، مع الاهتمام الخاص المولى للحالة التي يكون فيها المشتق قابلاً للإشتعال. تعتبر ACIMs كائنات هامة في دراسة الأنظمة الديناميكية، حيث توفر وسيلة لفهم السلوك طويل الأمد للمسارات والخصائص الإحصائية للنظام. تغطي الورقة مجموعة من المواضيع المهمة المتعلقة بـ ACIMs ، بما في ذلك شرط القابلية المحدودة ، وشرط التشويه ، وشرط التموضع ، وشرط شميت. كما يناقش المشغل بيرون-فروبينيوس ، الذي يلعب دورًا حاسمًا في وجود وخصائص ACIMs. الورقة تناقش أيضًا العلاقة بين ACIMs وأنظمة الديناميكا، مبرزة دور ACIMs في نظرية الإرغوديك. بشكل عام، تقدم هذه الورقة مرجع قيم للباحثين المهتمين بدراسة ACIMs وأهميتها في تحليل أنظمة الديناميكا ونظرية الإرغوديك.

1.Introduction

In the study of complex systems, dynamical systems theory provides a powerful framework for understanding their behavior over time. A fundamental concept in this theory is the notion of invariant measures, which capture the statistical properties of system trajectories. Among these measures, absolutely continuous invariant measures (ACIMs) have received significant attention due to their crucial role in characterizing the long-term dynamics of chaotic systems. This paper focuses on exploring ACIMs for piecewise expanding chaotic transformations in n-dimensional Euclidean space (R^n). These transformations possess the ability to stretch and fold the state space, giving rise to intricate and diverse dynamics. Our investigation specifically considers a scenario where the derivative of the

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transformation exhibits summable oscillations. This particular case presents intriguing challenges and necessitates a meticulous analysis to comprehend the resulting ACIMs. The significance of invariant measures in capturing general characteristics observed across different datasets has been highlighted in a recent validation and measurement study (2022) [1]. This understanding contributes to the comprehension of complex system behavior, enabling the analytical analysis of control parameters and the forecasting of future conditions based on underlying dynamic models. In the realm of dynamical systems theory, a self-consistent dynamical system refers to a system whose dynamics align with specific mathematical properties, including the existence of invariant measures, as discussed in a reference from 2019 [2]. In conjunction with ergodic theory (2016) [3], dynamical systems theory provides robust tools and concepts for comprehending the behavior and statistical properties of systems that evolve over time. These theories lay a solid mathematical foundation for studying a wide range of phenomena and find broad applications in various scientific and engineering disciplines. Another important concept pertains to random non-uniformly expanding maps, which are families of maps where each map possesses a distinct expansion rate, and the choice of the map is determined by a random process (2014) [4]. Such maps are often employed to model systems that exhibit a combination of deterministic and stochastic components. The study of random non-uniformly expanding maps involves investigating the existence and properties of absolutely continuous invariant measures. These measures assign zero probability to individual points and instead assign positive probabilities to sets of points, characterized by an integrable density function. M. Viana's "Lectures on Lyapunov Exponents" (2014) [5] focuses on quantifying the rate of exponential growth or decay of trajectories in dynamical systems through Lyapunov exponents. The book delves into the theory of Lyapunov exponents, their relationship with ACIMs, and the statistical properties of dynamical systems. In their Encyclopaedia of Complexity and Systems Science (2009/2013) [6], Kelso, James further highlights the importance of absolutely continuous invariant measures (ACIMs) in such research, particularly with regard to understanding behaviors over time. Kelso's subsequent studies [6] confirmed the value of ACIMs, especially in simulating chaotic processes or analyzing structured but challenging-to-quantify experimental data. "The Encyclopedia of Complexity and Systems Science, published in 2009 [7], is a comprehensive reference work that explores various aspects of complexity science and systems science. It offers an interdisciplinary overview of the field, covering fundamental principles, theories, methodologies, and applications related to complexity and systems science. In his book "Mathematics of Complexity and Dynamical Systems" (2011) [8], Robert Mayers emphasizes the importance of invariant measures in understanding the long-term behavior of dynamical systems. The book "Chaotic Billiards" by N. Chernov and R. Markarian [9] focuses on the dynamics of billiard systems characterized by particles rebounding off walls. It discusses the existence and properties of absolutely continuous invariant measures (ACIMs) in billiard systems and their connection to chaotic behavior. Another book, "Lyapunov Exponents and Smooth Ergodic Theory" by L. Barreira and Y. Pesin [10], offers a rigorous mathematical treatment of Lyapunov exponents and their significance in smooth ergodic theory. The book explores the interconnections between Lyapunov exponents, ACIMs, and the statistical behavior of dynamical systems.

The paper examines the relationship between metric entropy, ACIMs, and the ergodic properties of diffeomorphisms. C. Liverani's article on "Ergodicity properties of dynamical systems" [11] provides an overview of ergodic theory and its applications in the study of dynamical systems. The article discusses various ergodicity properties, including the existence and properties of ACIMs, and highlights connections to other mathematical areas. L.-S. Young's paper "What are SRB measures, and which dynamical systems have them?" [12] explores Sinai-Ruelle-Bowen (SRB) measures, a type of ACIMs associated with hyperbolic dynamical systems. The paper investigates the existence and properties of SRB measures and sheds light on the systems that possess them. V. Baladi's book "Positive Transfer Operators and Decay of Correlations" [13] delves into the theory of transfer operators and their applications in studying dynamical systems. The book covers the phenomenon of the decay of correlations, which is closely linked to the existence and properties of ACIMs, providing detailed mathematical analysis. Our research focuses on the exploration of absolutely continuous invariant measures (ACIMs) for piecewise expanding chaotic transformations in R^n . We investigate several crucial conditions necessary for understanding the existence and properties of ACIMs. One of these conditions is the boundedness condition, which ensures that the transformation does not stretch the state space to infinity. The boundedness condition is essential for the well-definedness of ACIMs, as it guarantees that the measures remain finite. We also examine the distortion condition, which quantifies the expansion and contraction rates of the transformation. By imposing bounds on these rates, we ensure that the system possesses a well-defined ACIM. This condition is closely tied to the concept of hyperbolicity, which characterizes the stretching and folding dynamics of chaotic systems.

Furthermore, we analyze the localization condition, which restricts the spread of trajectories in the state space. This condition ensures that the ACIM concentrates its mass on a finite region, providing a localized statistical description of the system's behavior. The localization of ACIMs is crucial for understanding the concentration of points and the emergence of coherent structures in chaotic systems. Another aspect of our investigation is the examination of Schmitt's condition [14], which imposes additional constraints on ACIMs. This condition ensures that the measures satisfy certain regularity properties, leading to a more refined understanding of the system's statistical behavior. By considering Schmitt's condition together with a condition dependent on the defining partition, we establish the existence of a finite number of ergodic ACIMs that exhibit desirable properties. To analyze the existence and properties of ACIMs, we delve into the theory of Perron-Frobenius operators. These operators play a fundamental role in the study of invariant measures, as they capture the evolution of probability densities under the action of the chaotic transformation. By investigating the properties of the Perron-Frobenius operator in the context of piecewise expanding chaotic transformations with summable oscillations of the derivative, we establish its constrictive nature. This result has profound implications, as it guarantees the existence of a finite number of ergodic ACIMs that satisfy Schmitt's condition and a condition dependent on the defining partition.

In summary, our paper provides a comprehensive investigation of ACIMs for piecewise expanding chaotic transformations in R^n, with a particular focus on scenarios where the derivative exhibits summable oscillations. By studying the boundedness, distortion, localization, and Schmitt's conditions, as well as the constrictiveness of the Perron-Frobenius operator, we establish the existence and properties of a finite number of ergodic ACIMs. This research contributes to our understanding of complex systems, their long-term behavior, and the statistical properties that govern their dynamics.

The article is structured as follows: Section 1 provides an introduction to absolutely continuous invariant measures (ACIMs), while Section 2 discusses the properties of piecewise expanding transformations. In Section 3, the article demonstrates how the transformation $\tau \in C^{1+\epsilon}$, which satisfies the three Rychlik criteria (distortion, localization, and boundedness), also satisfies certain conditions, including Schmitt's condition and the condition on the defining partition. Section 4 covers the construction of ACIMs, and the main result is presented in Section 5. Finally, Section 6 provides a discussion and conclusion, summarizing the findings of the article.

1.1. The concept of absolutely Continuous Invariant Measures (ACIMs)

In the field of dynamical systems, ACIMs are a crucial notion that enable the analysis of long-term behavior. These measures represent probability distributions that remain unchanged when subjected to a given transformation, allowing researchers to study the statistical properties of the system and make predictions about its future behavior.

1.2. Expanding Transformations in $\mathbb{R}^n$

Focuses on expanding transformations in R^n, which are an important class of systems in the realm of dynamical systems. These transformations cause the distances between points in the phase space to increase when the transformation is applied. Because of their tendency to exhibit chaotic behavior, they are particularly intriguing and have practical applications in various scientific fields, such as fluid dynamics, celestial mechanics, and population dynamics. The section also introduces notation for transformations, where the Jacobian matrix of a transformation is denoted by $\mathcal{J}$, and the absolute value of its determinant is denoted by $|\mathcal{J}|$.

1.3 Definition. presents the definition of an α -expanding transformation. Consider a finite partition of the phase space Ω, denoted by $\{P_i, P_2, \ldots, P_m\}$, and let $\tau: \Omega \to \Omega$ preserves this partition. That is, $\tau \equiv \tau_{|P_i}$. We say that τ is α-expanding if the 2-norm of the inverse Jacobian matrix of τ evaluated at any point in each partition element P_i is less than α^{-1}, where α is a constant greater than 1. In other words for $i = 1, 2, \ldots, m$ and $\alpha > 1$, we have:

$$\|J_{\tau_i^{-1}}\|_2 < \alpha^{-1},$$

2. Piecewise Expanding Transformations

2.1. Definition of piecewise expanding transformations

Piecewise expanding transformations are a type of mathematical transformation that is commonly used in the study of dynamical systems and chaotic behavior. A piecewise expanding transformation is a map that is defined on a domain that is partitioned into regions, such that the map is expanding [15] (i.e., it stretches the distances between points) within each region. The map may also be discontinuous at the boundaries between regions.

More formally, let $T: X \to X$ Where X is metric space. T is said to be a piecewise expanding transformation if there exists a partition of X into finitely many disjoint open sets $\{U_1, U_2, \ldots, U_n\}$ such that:

1. T is continuously differentiable on each U_i , and the derivative $|T'| > 1, \forall\, U_i$

2. The boundary of each U_i has zero Lebesgue measure.

3. T is discontinuous at the boundaries between the $U_i's$.

This means that T stretches distances between points within each open set U_i, but it does not stretch distances across the boundaries between the $U_i's$. The discontinuity at the boundaries allows for the possibility of chaotic behavior, as small changes in the initial conditions can lead to very different trajectories within different regions.

2.2. Properties of piecewise expanding transformations

Piecewise expanding transformations have several important properties that are relevant to their study in dynamical systems and chaos theory. Here are some of the main properties:

1. Expansivity: Within each open set U_i of the partition, the transformation T expands distances between points. More precisely, the derivative $|T'|$ is uniformly greater than 1 on each U_i. This means that nearby points in U_i. will be pushed farther apart by the action of T, which is an essential ingredient in generating chaotic behavior.

2.Discontinuity: Piecewise expanding transformations are discontinuous at the boundaries between the open sets U_i.. This allows for the possibility of chaotic behavior, as small perturbations near the boundaries can cause trajectories to diverge rapidly in different directions.

3. Bounded distortion: Although distances between points within each U_i. are expanded by T, the expansion is controlled by a uniform bound on the derivative $|T'|$. This means that T does not distort distances too much, and nearby points remain approximately close to each other even after several iterations of T.

4. Invariant measures: Piecewise expanding transformations often possess invariant measures, which are probability measures that are preserved by the action of T. These measures provide a way to understand the long-term statistical behavior of the system, and they are often used to compute statistical properties such as fractal dimensions and Lyapunov exponents.

5.Ergodicity: Many piecewise expanding transformations are ergodic, which means that they exhibit a strong form of statistical mixing. In particular, if a transformation is ergodic with respect to an invariant measure, then almost every trajectory in the system visits every part of the phase space with the same frequency, regardless of the initial conditions. This property is important for understanding the long-term behavior of chaotic systems.

2.3. Summable Oscillations of Derivative

One essential feature of piecewise expanding chaotic transformations is the summable oscillations exhibited by their derivative. This property guarantees that the transformation's derivative has a uniformly bounded variation, which is crucial for the existence and uniqueness of ACIMs in these systems. The summability of the oscillations allows researchers to examine the statistical characteristics of the system and extract valuable insights into its long-term behavior.

2.4. Definition. We consider an open and bounded subset Ω, of $\mathbb{R}^{\boldsymbol{n}}$ with a piecewise C^2 boundary. A partition $\mathcal{P} = \{P_1, P_2, \ldots, P_m\}$, where m is a finite number, is said to be smooth if each $P_i, i = 1,2, \ldots, m$ has a boundary that is piecewise C^1.

2.5. Remark. Let $\mathcal{P} = \mathcal{P}^{(1)} = \{P\}_{i=1}^{m}$ be a smooth partition of an open and bounded subset Ω, of R^n, and

$$\mathcal{P}^{(k)} = \bigvee_{k=0}^{k-1} \tau^{-j}(\mathcal{P})$$

$$= \left\{P_{i_1} \cap \tau^{(-1)}\left(P_{i_2}\right) \cap \ldots . \tau^{-j+1}\left(P_{i_j}\right) : P_{i_k} \in \mathcal{P}\ \ for\ 1 \le k \le j\right\}$$

let τ be a piecewise expanding transformation. We define, $P^{(k)}$ as the partition obtained by successively precomposing P with the inverse of τ, up to k times. The sets in $P^{(k)}$ are defined as intersections of k sets from P, possibly permuted by the inverse of τ.

We also define $I_j = \left\{i: P_i \in \mathcal{P}^{(k)}\right\}$. For a fix $i \in I_k$ Here are the following:

$$\max_{i \in I_j} \lambda_n(P_i) \le \alpha^{-k},$$

where λ_n denotes the Lebesgue measure on R^n. The Lebesgue measure is a mathematical concept used to assign a measure or size to subsets of Euclidean space. It is named after the French mathematician Henri Lebesgue [16], who developed this theory in the early 20th century as an extension of the concept of length or area. The Lebesgue measure is defined as a class of subsets of Euclidean space, such as the real line, the plane, or higher-dimensional spaces. The basic idea is to assign a non-negative number to each subset of the space, which represents its "size" or "volume" in an intuitive sense.). This implies that the size of the largest element in $P^{(j)}$ converges to zero as $j \to \infty$. Consequently, the σ-algebra generated by the nested sequence of partitions $\cup_{j \ge 1} \mathcal{P}^{(j)}$ coincides with the Borel σ-algebra $\mathcal{B}$ of Ω.

Moreover, for $\boldsymbol{i} \in \boldsymbol{I_k}$, we define the

$$\operatorname*{osc}_{P_i} \mathcal{J}_\tau = \max_{P_i} \mathcal{J}_\tau - \min_{P\_i} \mathcal{J}_\tau,$$

Where $\mathcal{J}_\tau$ is the Jacobian determinant of τ. We then define

$$\Delta_k = \max_{i \in \mathcal{J}_k} \operatorname*{osc}_{P_i} \mathcal{J}_\tau\ ,$$

To measure the maximum oscillation of $\mathcal{J}_\tau$ on the sets in $\mathcal{P}^{(k)}$. Piecewise Expanding Chaotic Transformations are known to exhibit chaotic behaviour, which enables us to study invariant measures and the long-term behavior of systems described by Absolutely Continuous Invariant Measures (ACIM).

3. Meeting the conditions for ACIMs

The purpose of this section is to explore how piecewise expanding transformations can meet the conditions required for ACIMs to exist and be unique. Additionally, we discuss Schmitt's condition and the Rychlik criteria (distortion, localization, and boundedness) and provide a lemma and remarks on how these conditions can be satisfied.

3.1. Definition. This subsection introduces the concept of an ϵ-transformation, which is a piecewise expanding transformation that satisfies Schmitt's Condition. The condition requires that the sum of maximum oscillations of Jacobian determinants, denoted by Δ_k, is finite. Moreover, there exists a constant Λ such that the Jacobian determinant of the transformation is bounded above by Λ on the entire phase space Ω.

$$\textstyle\sum_{k \ge 1} \Delta_k = \Delta < +\infty.$$

The condition of Schmitt suggests that there is a constant Λ, such that

$$J_\tau(x) \le \Lambda, x \in \Omega.$$

Prior to continuing to the following Remark 3.4, let

$$\mathcal{P}(B) = \{A \in \mathcal{P} : \lambda(A \cap B) > 0\},$$

as the collection of elements in the partition $\mathcal{P}$ that have non-zero intersection with a Borel set B.

3.2. Remark: It was demonstrated by Rychlik [17] that these Schmitt conditions have a fixed point. $\mathcal{P}_\tau$. In our context, we assume the following operator for the three conditions:

I. For the Distortion condition: There exists $\delta > 0$ such that for any $k \geq 1$ and any $B \in \mathcal{P}^{(k)}$, we have
$$\sup_B \mathcal{J}_{\tau^k}(\tau_i^{-k}(x)) \leq \delta \inf_B \mathcal{J}_{\tau^k}(\tau_i^{-k}(x)).$$

II. For the Localization condition: There exist $\eta > 0$ and $0 < \rho < 1$ such that for any $i \geq 1$ and any $B \in \mathcal{P}^{(i)}$:
$$\lambda_n\left(\tau^i(B)\right) < \eta \Rightarrow \sum_{A\in P(\tau^i(B))} \sup_A \mathcal{J}_\tau(\tau_i^{-k}(x)) \leq \rho.$$

III. For the Boundedness condition:
$$\gamma = \sum_{B\in P} \sup_B \mathcal{J}_\tau(\tau^{-1}(x)) < \infty.$$

The Localization condition holds if $\tau^i B$ is small in measure and does not intersect too many elements of $\mathcal{P}$ with a large value of $\mathcal{J}_\tau(\tau^{-1}(x))$. These conditions ensure that the transformation satisfies Schmitt's Condition. In particular, the Distortion Condition guarantees that the expansion rate is bounded, the Localization Condition ensures that the transformation is well-behaved in a local sense, and the Boundedness Condition guarantees that the ACIM is supported on a compact set.

3.3. Remark: Suppose that $x, y \in B \in \mathcal{P}^{(\nu+k)}(\tau^i(y), \tau^i(x) \in \tau^{-1}B \in \mathcal{P}^{(\nu+k-i)})$, then:
$$\mathcal{J}_\tau\left(\tau^i(y)\right) - \mathcal{J}_\tau\left(\tau^i(x)\right) \leq \max_{\tau^{-1}B\in P^{\nu+k-i}}\left(\max_{\tau^{-1}B}\mathcal{J}_\tau\left(\tau^i(y)\right) - \min_{\tau^{-1}B}\mathcal{J}_\tau(\tau^i(x))\right) = \max_{\tau^{-1}B\in P^{\wedge}\{(\nu+k-i)\}} \operatorname{osc}_{\tau^{-1}B} \mathcal{J}_\tau = \Delta_{\nu+k-i}$$

This is equal to the maximum oscillation of the Jacobian determinant of τ on the set $\tau^{-1}B$, denoted by $\Delta_{\nu+k-i}$. In other words, Remark 3.3 highlights a relationship between the maximum difference in the Jacobian determinant of τ at points $\tau^i(y)$ and $\tau^i(x)$ and the maximum oscillation of the Jacobian determinant on the set $\tau^{-1}B$. This relationship is useful in analysing the behavior of expanding transformations and can be used to establish conditions for the existence of Absolutely Continuous Invariant Measures (ACIMs).

3.4. Lemma*:* Show that if τ is an ε-transformation, then the distortion conditions are satisfied.

Proof. To prove this, consider any $x, y \in B \in \mathcal{P}^{(\nu+k)}(\tau^i(y), \tau^i(x) \in \tau^{-1}B \in \mathcal{P}^{(\nu+k-i)})$.

Using the definition of an ε-transformation and Remark 3.5, we obtain:
$$\frac{\mathcal{J}_{\tau^k}(y)}{\mathcal{J}_{\tau^k}(x)} = \prod_{i=0}^{k-1}\frac{\mathcal{J}_\tau(\tau^i(y))}{\mathcal{J}_\tau(\tau^i(x))} \leq \prod_{i=0}^{k-1} exp\left(\frac{\mathcal{J}_\tau\left(\tau^i(y)\right)-\mathcal{J}_\tau(\tau^i(x))}{\mathcal{J}_\tau(\tau^i(x))}\right).$$

According to Remark 3.3 and the definition of an ε- transformation, we obtain:
$$\frac{\mathcal{J}_{\tau^k}(y)}{\mathcal{J}_{\tau^k}(x)} \leq \prod_{i=0}^{k-1} exp\left(\frac{1}{\alpha}\Delta_{\nu+k-i}\right) = exp\left(\frac{1}{\alpha}\sum_{i=0}^{k-1}\Delta_{\nu+k-i}\right) = exp\left(\frac{1}{\alpha}\sum_{i=0}^{k}\Delta_{\nu+i}\right).$$

Letting $\delta = exp\left(\frac{1}{\alpha}\sum_{i=1}^{k}\Delta_{\nu+i}\right)$ complete the proof.

The distortion condition is critical for establishing the existence of Absolutely Continuous Invariant Measures (ACIMs) for expanding transformations. It relates the local behavior of the transformation to its global behavior and is essential for understanding the long-term behavior of the system. Lemma 3.4 provides a fundamental result that helps to establish the distortion condition for a broad class of expanding transformations.

3.5. Lemma: If τ is an ε-transformation, then the boundedness conditions are satisfied.

Proof. We have $\gamma = \sum_{B\in P}\sup_B \mathcal{J}_\tau(\tau^{-1}(x)) = \#B\frac{1}{\mathcal{J}_\tau(x)} \leq \#B\alpha^{-1} < \infty$, where $\#B$ denotes the number of elements in the partition P containing x. This inequality follows from the definition of an ε-transformation and the fact that J_τ is bounded away from zero. Therefore, the boundedness condition is satisfied.

This result is important for establishing the existence of Absolutely Continuous Invariant Measures (ACIMs) for expanding transformations, as it ensures that the ACIM is supported on a compact set. Lemma 3.5 shows that the boundedness condition holds for a broad class of expanding transformations, namely ε-transformations.

3.6. Remark: Introduces a way to measure the stability of an expanding transformation within a bounded region of the phase space.

Let $L_i(x_1, x_2, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$, be represented by a straight line that is parallel to the i −th axis and has all the coordinates, with the exception of the i −th fixed and equal to $x_1, x_2, \dots \dots x_{i-1}, , , , , , , , x_n$. Let
$$R_i^{(j)}(x_1, x_2, \dots, x_{i-1}, x_{i+1}, \dots, x_n) = \#\{P_i : P_i \in \mathcal{P}^{(j)}, P_i \cap L_i(x_1, x_2, \dots, x_{i-1}, x_{i+1}, \dots, x_n) \neq 0\}$$
For any $1 \leq i \leq n$, let
$$\mathcal{R}_i^{(j)} = \sup_{\substack{x_k\\1\leq k\leq n\\k\neq i}} R_i^{(j)}(x_1, x_2, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$$
$$\overline{R_\iota^{(J)}} = \prod_{\substack{k=1\\k\neq i}}^{n}\mathcal{R}_k^{(j)}$$
$$R^{(j)} = \max_{1\leq i\leq n}\acute{\mathcal{R}_\iota^{(J)}}$$
$$\mathcal{R} = \max_j \mathcal{R}^{(j)}$$

In this case, the expanding transformation is well-behaved and demonstrates stable behavior within a bounded region of the phase space. Stability is essential to understanding ACIM's long-term behavior and establishing its existence. By measuring the stability of an expanding transformation through the parameter R, Remark 3.6 provides a useful tool for analyzing the behavior of ACIMs and for identifying conditions under which they exist.

3.7. Lemma: If τ is an ε-transformation, and $\alpha > 2\mathcal{R}$, then the localization condition is satisfied.

Proof. Take $\eta > 0$ such that $\eta^{\frac{1}{n}} < min_{i\in I_1} diam\, P_i$ and $0 < \rho < 1$, such that
$$0 < \frac{2\mathcal{R}}{\alpha} < \rho < 1.$$
Since $min\, diam_{i\in I_1} P_i > \eta^{\frac{1}{n}}$, for any $B \in \mathcal{P}^i, \lambda_n\left(\tau^i(B)\right) < \eta$, means that at least one of the widths of $\tau^i(B)$ is smaller than $\eta^{\frac{1}{n}} < min_{i\in I_1} diam P_i$, and the number of $\mathcal{P}\left(\tau^i(B)\right)$ is at most $2\mathcal{R}$. Therefore
$$\sum_{A\in P(\tau^i(B))}\sup_A \mathcal{J}_\tau(\tau^{-1}(x)) \leq \frac{2\mathcal{R}}{\propto} < \rho.$$

When the localization condition is met, the expanding transformation is likely to be rather stable in the phase space. Understanding the system's long-term behavior and ensuring ACIM's existence depends on its stability. Lemma 3.7 provides a condition for the localization of an expanding transformation, which is a crucial step in establishing the existence of ACIMs. By ensuring that the localization condition is satisfied, we can guarantee that the expanding transformation is stable in a bounded region of the phase space, allowing for a better understanding of its long-term behavior.

3.8. Definition. For any $t > 0$ let $\nu(t)$ be the largest ν such that
$$\min_{P\in\mathcal{P}^{(\nu)}} \lambda_n(P) > t$$
The term is defined by $\omega_\tau(t) = max\{t, exp\left(\frac{2}{\alpha}\sum_{k\geq\nu}\Delta_k\right) - 1\}$.

Notice that: $1 - exp\left(\frac{-2}{\alpha}\sum_{k\geq\nu}\Delta_k\right) \leq exp\left(\frac{2}{\alpha}\sum_{k\geq\nu}\Delta_k\right) - 1$.

Definition 3.8 provides a way to measure the rate of decay of the tails of the distribution of the expanding transformation. The parameter $\omega_\tau(t)$ captures the exponential decay of the tails and is related to the distortion and boundedness conditions. The quantity $\nu(t)$ determines the scale at which the tails decay, and is related to the localization condition. By understanding the behavior of $\omega_\tau(t)$ and $\nu(t)$, we can gain insights into the long-term behavior of the system and establish the existence of Absolutely Continuous Invariant Measures (ACIMs).

3.9. Lemma: If τ is an ε-transformation, then; for any $t > 0, B \in P^{(j)}$ and $x, x+t \in \tau^i(B), i \le j$:

$$\left|\frac{\mathcal{J}_{\tau^i}(x)}{\mathcal{J}_{\tau^i}(x+t)} - 1\right| \le exp\left(\frac{2}{\alpha}\sum_{k=1}^{j}\Delta_{\nu(t)+k}\right) - 1.$$

Where $\nu(t)$ is defined in Definition 3.8.

Proof. According to the definition of $\nu(t), x$ and $x+t$ are in the same or two neighbouring sets of $\mathcal{P}^{(\nu(t))}$. Additionally, as τ^i is a one-to-one transformation between B and $\tau^i(B)$, points $\tau^{-i}(x)$ and $\tau^{-i}(x+t)$ in B are likewise found in the same or two neighbouring sets of $\mathcal{P}^{(\nu(t)+i)}$. Using Lemma 3.4, we obtain:

$$exp\left(\frac{-2}{\alpha}\sum_{i=k}^{j}\Delta_{\nu(t)+k}\right) \le \frac{\mathcal{J}_{\tau^i}(x)}{\mathcal{J}_{\tau^i}(x+t)} \le exp\left(\frac{2}{\alpha}\sum_{k=1}^{j}\Delta_{\nu(t)+k}\right).$$

That makes the proof complete. Lemma 3.9 provides an estimate of the distortion of the expanding transformation between two nearby points in the same partition element. The bound on the distortion is characterized by the parameter $\nu(t)$ and the distortion constant α. By understanding the behavior of the distortion, we can gain insights into the long-term behavior of the system and establish the existence of Absolutely Continuous Invariant Measures (ACIMs).

4. Construction of ACIM for Expanding Transformations Satisfying Schmitt's Condition

In this section, we describe the techniques and tools used to construct Absolutely Continuous Invariant Measures (ACIMs) for expanding transformations satisfying Schmitt's condition. These techniques often rely on mathematical concepts and tools such as the Perron-Frobenius operator and numerical methods such as Monte Carlo sampling and Markov chain Monte Carlo (MCMC) [18]. However, there are also challenges and limitations involved in constructing ACIMs, which we will discuss.

4.1. Perron-Frobenius Operator

For expanding transformations satisfying Schmitt's condition, the Perron-Frobenius operator is an effective tool for constructing ACIMs. This operator allows researchers to locate and analyse the invariant measures associated with the system by mapping a given function to its future behavior after the transformation.

4.2. Techniques and Algorithms

Various methods and algorithms have been developed for constructing ACIMs for expanding transformations in $\mathbb{R}^{\boldsymbol{n}}$ that satisfy Schmitt's condition. These techniques often rely on mathematical concepts and tools such as the Perron-Frobenius operator. The use of the Perron-Frobenius operator can significantly impact the ergodic properties of the system, including ergodicity, mixing, and the Central Limit Theorem. The analysis of the Perron-Frobenius operator in the presence of an ACIM can yield valuable insights into these properties of the system. However, constructing ACIMs can also be challenging and computationally intensive, requiring the use of numerical methods such as Monte Carlo sampling and MCMC. Furthermore, the existence and uniqueness of ACIMs are not guaranteed in all cases, depending on the properties of the expanding transformation.

4.3. Remark: The Perron-Frobenius operator is the main foundation of this paper. It is denoted as $P_\tau: L^1(\Omega) \to L^1(\Omega)$ For a function f in $L^1(\Omega)$, P_τ f is given by the formula:

$$P_\tau f(x) = \sum_{i\in I_1}\frac{f\left(\tau_i^{-1}(x)\right)}{\mathcal{J}_\tau\left(\tau_i^{-1}(x)\right)}, f \in L^1(\Omega).$$

where τ is a measure-preserving transformation of Ω, I_1 is the index set of the τ-preimages of x, and J_τ is the Jacobian determinant of τ.

The Perron-Frobenius operator is a Markov operator [18], which means that it satisfies the following properties:

(A) $P_\tau f \ge 0, f \ge 0$, for all f in $L^1(\Omega)$.

(B) $\|P_\tau f\|_{L^1(\Omega)}, for f \ge 0$, for all non-negative functions f in $L^1(\Omega)$.

It can also be shown that a function $f \in L^1(\Omega)$ is a density of a τ-invariant measure if and only if $P_\tau f = f$ and

$$\int_\Omega f \cdot g \ o\ \pi^k d\lambda_n = \int_\Omega P^k f \cdot g\ d\lambda_n\ ,$$

for all g in $L^\infty(\Omega)$, where π^k is the kth power of the Perron-Frobenius operator and λ_n is the normalized Lebesgue measure on Ω

4.4. Definition. The Markov operator $P_\tau: L^1(\Omega) \to L^1(\Omega)$ is said to be extremely constrictive if there exists a compact set $F \subset L^1(\Omega)$ such that for any non-negative function $f \in L^1(\Omega)$ with $f \ge 0$ and $\|f\|_{L^1(\Omega)} = 1$, hence:

$dis(P_\tau f, F) = 0, in\ L^1(\Omega)\ \ metric$. (1).

The main goal of this study is to show that for a ε-transformation defined on a partition satisfying a certain condition, the operator P_τ is strongly constrictive on $L^1(\Omega)$. The concept of extreme constrictivity is important for understanding the long-term behavior of expanding transformations and for establishing the existence of ACIMs.

5. The Main Results

This section presents the main findings of the article, which focus on a new theorem or result related to ACIMs for piecewise expanding chaotic transformations in $\mathbb{R}^{\boldsymbol{n}}$ with summable oscillations of derivative. The section also explores the potential implications of the result and its practical applications. The primary outcomes of the study are based on the definitions of an exclusive isomorphism to L^1, as well as two subspaces of $L^1 = L^1\ (\Omega, B, \lambda)$, which facilitates the creation of an appropriate subset F of $L^1(\Omega)$. These definitions are firmly grounded in Rychlik's theories [9]. Furthermore, a space FS_τ of formal series is introduced, defined as follows:

$$\mathcal{F}S_\tau = \left\{F_\beta | F_\beta = \sum_{\substack{1\le i\le j\\ B\in\mathcal{P}^{(j)}}} \beta_{(i,j,B)} P_\tau^i \chi B, \beta_{(i,j,B)} \in \mathbb{R}^n\right\}\ and$$

$$\sum_{\substack{1\le i\le j\\ B\in\mathcal{P}^{(j)}}} \max_{1\le k\le n} |\beta_{(i,j,B)}^k| \|P_\tau^i \chi B\|_{L^\infty(\Omega)} < \infty.$$

Here, $\beta_{(i,j,B)} = \left(\beta_{(i,j,B)}^1, \beta_{(i,j,B)}^2, , \ldots\ldots, \beta_{(i,j,B)}^n\right)$. The norm in FS_τ defined as:

$$\|F_\beta\|_{FS_\tau} = \sum_{(i,j,B)\in\mathcal{A}_\tau} \max_{1\le k\le n} |\beta_{(i,j,B)}^k|\ \left\|P_\tau^i(\chi B)\right\|_{L^\infty(\Omega)}.$$

The space FS_τ is a Banach space. We also define the operator P_τ on FS_τ and a set BF_τ of bounded functions in $L^1(\Omega)$ as follows:

For any $F_\beta = \sum_{\substack{1\le i\le j\\ B\in\mathcal{P}^{(j)}}} \beta_{(i,j,B)} P_\tau^i(\chi B) \in FS_\tau$, has

$$P_\tau F_\beta = \sum_{\substack{1\le i\le j\\ B\in\mathcal{P}^{(j)}}} \beta_{(i,j,B)} P_\tau\left(P_\tau^i(\chi B)\right). \qquad (2)$$

Any $F_\beta \in FS_\tau$ defines a bounded function as

$$f_\beta = \Phi(F_\beta) = \sum_{\substack{1\le i\le j\\ B\in\mathcal{P}^{(j)}}} \beta_{(i,j,B)}\ P_\tau^i(\chi B),$$

Let

$$BF_\tau = \{f \in L^1(\Omega): f = f_\beta = \Phi(F_\beta), F_\beta \in FS_\tau\}.$$

The norm in $BF_\tau$is defined as;

$$\|\Phi(F_\alpha)\|_{BF_\tau} = \inf_{F_\beta:\ \Phi(F_\beta)=\Phi(F_\alpha) a.e} \|F_\beta\|_{FS_\tau}.$$

$BF_\tau \subset L^\infty(\Omega)$ and $\|f_\beta\|_\infty \le \|f_\beta\|_{BF_\tau}$ exist. Furthermore, it is straightforward to demonstrate that $F_{\beta^{(k)}} \to F_{\beta^0}$ in FS_τ implies pointwise convergence of $f_{\beta^{(k)}}$ to f_{β^0} in $L^1(\Omega)$. According to both facts, the operator Φ form FS_τ onto $L^1(\Omega)$ is continuous. Due to the face that $BF_\tau = \frac{FS_\tau}{Ker(\Phi)}$, BF_τ with the norm $\|.\|_{BF_\tau}$ is a Banach space. Finally, any function $f \in L^1(\Omega)$, has an integral modulus of continuity that is

$$W_f(t) = \int_{\Omega} |f(x+t) - f(x)| d\lambda_n(x).$$

5.1. Remark: It is worth noting that the integral modulus of continuity of the sum of two functions f and g is bounded by the sum of their individual moduli of continuity. That is;

$$W_{f+g}(t) = \int_{\Omega} |(f+g)(x+t) - (f+g)(x)| \, d\lambda_n(x)$$
$$= \int_{\Omega} |f(x+t) + g(x+t) - f(x) - g(x)| \, d\lambda_n(x)$$
$$\leq \int_{\Omega} |f(x+t) - f(x)| d\lambda_n(x) + \int_{\Omega} |g(x+t) - g(x)| d\lambda_n(x)$$
$$= W_f(t) + W_g(t).$$

Let $\omega(.)$be a continuity modulus, which is a non-negative increasing function with $\omega(t) = 0$. We then introduce a subspace of $L^1(\Omega)$, denoted by $MC\omega$, consisting of functions with a bounded modulus of continuity. This is defined as:

$$MC\omega = \{f \in L^{\infty}(\Omega) : W_f(t) \leq K . \omega(t), for\ some\ K > 0 \}$$

Here, $\omega(.)$ is a continuity modulus, i.e. a non-negative increasing function with $\omega(t) = 0$. We define a norm on $MC\omega$ as follows:

$$\|f\|_{MC\omega} = max\, \{\|f\|_{L^{\infty}(\Omega)}, K_f\},$$

Where $K_f = inf\,\{K : W_f(t) \leq K . \omega(t)\}$. $MC\omega$ is a Banach space. In the following, we demonstrate some of the space properties of FS_{τ}, BF_{τ} and $MC\omega$.

We will use the compact subset of $L^1(\Omega)$ that is defined by the following two propositions as F.

5.2. Proposition. Every bounded subset of MC_{ω} is precompact in $L^1(\Omega)$. Hence, MC_{ω} represents the space of probability measures on the infinite sequence space Ω equipped with the topology of weak convergence.

Proof. This result is obtained from Kolmogorov's theorem [19], which is a classical and fundamental theorem in probability theory. The theorem establishes a connection between the convergence of a sequence of random variables and its convergence in L^1. Specifically, the theorem states that a sequence of random variables that are uniformly integrable and converge in probability must converge in L^1. Since every element of MC_{ω} is a probability measure, it follows that every bounded subset of MC_{ω} satisfies the conditions of Kolmogorov's theorem [19]. Therefore, every bounded subset of MC_{ω} is precompact in $L^1(\Omega)$. Which implies that MC_{ω} represents the space of probability measures on Ω equipped with the topology of weak convergence.

5.3. Proposition. Let f_k be a sequence in MC_{ω} such that $\|f_k\|_{MC\omega} \leq M$, for all $k = 1,2,\ldots$.

If $f_k$converges to f in $L^1(\Omega)$, then $f \in MC_{\omega}$and

$$\|f\|_{MC\omega} \leq lim\ \|f_k\|_{MC\omega} \leq M$$

Proof. We first note that since $f_k \to f$ in $L^1(\Omega)$, there exists a subsequence f_{k_i} such that f_{k_i} converges to f almost everywhere. Therefore, f_k converges to f almost everywhere. It follows that

$$\|f\|_{L^{\infty}(\Omega)} \leq lim\ \|f_k\|_{L^{\infty}(\Omega)} \leq M.$$

According to the definition of $\|.\|_{MC\omega}$, $\|f\|_{L^{\infty}(\Omega)} \leq \|f_k\|_{MC\omega}$, applying this along with $\|f_k\|_{MC\omega} \leq M$, for all k, in the previous equation, we get:

$$\|f\|_{L^{\infty}(\Omega)} \leq M.$$

To prove that f belongs to $MC\omega$, we use the definition of $W_{f(t)}$ which is the integral modulus of continuity of f. By the assumption that f_k converges to f in $L^1(\Omega)$, we have the following:

$$W_f(t) = \int_{\Omega} |f(x+t) - f(x)| \, d\lambda_n(x) \quad (3)$$
$$= \int_{\Omega} |f_k(x+t) - f_k(x)| \, d\lambda_n(x)$$
$$= W_{f_k}(t)$$

Since $f_k \in MC_{\omega}$ therefore,

$$W_{f_k}(t) \leq lim\, K_{f_k}\omega(t),$$

For some constant K_{f_k} and for all t. Using the definition of $\|.\|_{MC_{\omega}}$, $K_{f_k} \leq \|f_k\|_{MC\omega} \leq M$. Substituting this inequality into the previous inequality, we get:

$$W_{f_k}(t) \leq M\omega(t)\ \ \forall\, k \quad (4)$$

By using (4) in (3) we obtain:

$$W_f(t) \leq M\omega(t) = M\omega(t).$$

Therefore $f \in MC_{\omega}$ and

$$\|f\|_{MC_{\omega}} = max\{\|f\|_{L^{\infty}(\Omega)}, K_f\}$$

As a result, we can use Equations (3) and (4) we obtain:

$$\|f\|_{MC_{\omega}} = max\{lim\, \|f\|_{L^{\infty}(\Omega)},\ K_f\} \leq M$$

This completes the proof, which involves using Equations (3) and (4) that rely on the definition of K_f and $W_f(t)$.

5.4. Corollary: The set

$$\left\{f \in MC_{\omega} : \|f\|_{MC_{\omega}} \leq \frac{3\gamma \frac{\delta}{\eta}}{1-\rho}\right\}$$

Is a compact subset of $L^1(\Omega)$.

Proof. The proof follows directly from Propositions 5.2 and Proposition 5.3, which establish that every bounded subset $MC_{\omega}$is precompact in $L^1(\Omega)$ and that the MC_{ω} norm of a sequence of functions that converges to a function in $L^1(\Omega)$ is bounded by the limit of the MC_{ω} norms of the sequence. Since $\mathcal{F}$ is defined as a subset of MC_{ω} with a bounded MC_{ω} norm, $\mathcal{F}$ is a bounded subset of MC_{ω}. Therefore, by Proposition 5.2. $\mathcal{F}$ is precompact in $L^1(\Omega)$. Furthermore, by Proposition 5.3, if a sequence of functions in $\mathcal{F}$converges to a function in $L^1(\Omega)$, then the limit function also belongs to MC_{ω} and has a bounded MC_{ω} norm. Therefore, $\mathcal{F}$ is closed in $L^1(\Omega)$.

Since $\mathcal{F}$ is both precompact and closed in $L^1(\Omega)$, it is compact. Thus, we have shown that $\mathcal{F}$ is a compact subset of $L^1(\Omega)$.

5.5. Lemma: For a fixed positive number, for any, $1 \leq i \leq j$, and an $B \in P^{(j)}$, then

$$W_{P_{\tau}^{i}XB}(t) \leq 3\|P_{\tau}^{i}XB\|_{L^{\infty}(\tau^{i}(B))}\omega_{\tau}(t). \quad (5)$$

where $P_{\tau}^{i}XB$ denotes the i-th iterate of the Perron-Frobenius operator applied to the function XB, and W is the Wasserstein distance between the probability measures induced by the functions.

Proof. To prove this lemma, we first use the definition of the Perron-Frobenius operator to express $P_{\tau}^{i}XB$ in terms of the function $X_{\tau}^{i}(B)$ and the Jacobian J_{τ}^{i}. Then, we upper bound the Wasserstein distance by integrating the absolute difference of $P_{\tau}^{i}XB$ evaluated at two points. By simplifying the integrand and applying the reverse triangle inequality, we obtain a bound on the integrand in terms of the absolute values of the function and its Jacobian. We then apply Lemma 3.5 with appropriate choices of A and g to get the desired upper bound on the integral, which leads to the desired inequality (5).

$$P_{\tau}^{i}XB(x) = \frac{X_{\tau^{i}(B)}(x)}{\mathcal{J}_{\tau^{i}}(x)}$$

An upper bound for $W_{P_{\tau XB}^{i}}(t)$ can be obtained by,

$$W_{P_{\tau}^{i}XB}(t) = \int_{\Omega} \left|P_{\tau}^{i}XB(x+t) - P_{\tau}^{i}XB(x)\right| d\lambda_n$$
$$= \int_{\Omega} \left|\frac{X_{\tau^{i}(B)}(x+t)}{\mathcal{J}_{\tau^{i}}(x+t)} - \frac{X_{\tau^{i}(B)}(x)}{\mathcal{J}_{\tau^{i}}(x)}\right| d\lambda_n$$
$$= \int_{\tau^{i}(B)} \left|\frac{X_{\tau^{i}(B)}(x+t)}{\mathcal{J}_{\tau^{i}}(x+t)} - \frac{X_{\tau^{i}(B)}(x)}{\mathcal{J}_{\tau^{i}}(x)}\right| d\lambda_n \quad (6)$$

Note that, (6) is equivalent to:

$$W_{P_{\tau}^{i}XB}(t) = \int_{\tau^{i}(B)} \left|P_{\tau}^{i}XB(x+t) - P_{\tau}^{i}XB(x)\right| d\lambda_n$$
$$\leq \int_{\tau^{i}(B)} \left|P_{\tau}^{i}XB(x+t)\right| d\lambda_n + \int_{\tau^{i}(B)} \left|P_{\tau}^{i}XB(x)\right| d\lambda_n$$

$$\le \|P_\tau^i XB\|_{L^\infty(\tau^i(B))}\lambda_n(\tau^i(B)) + \|P_\tau^i XB\|_{L^\infty(\tau^i(B))}\lambda(\tau^i(B)) = 2\,\lambda_n(\tau^i(B))\,\|P_\tau^i XB\|_{L^\infty(\tau^i(B))}$$

If $max\ diam\left(\tau^i(B)\right) < t$, then $W_{P_\tau^i XB}(t) \le 2t\,\|P_\tau^i XB\|_{L^\infty(\tau^i(B))}$.
Next, we observe that;

$$W_{P_{\tau XB}^i}(t) = \int_{\tau^i(B)} \left|\frac{1}{\mathcal{J}_{\tau^i}(x+t)} - \frac{1}{\mathcal{J}_{\tau^i}(x)}\right| d\lambda_n$$

$$= \int_{\tau^i(B)} \left|\mathcal{J}_{\tau^i}\left(\tau^{-i}(x+t)\right) - \mathcal{J}_{\tau^i}\left(\tau^{-i}(x)\right)\right| d\lambda_n$$

$$= \int_{\tau^i(B)} \mathcal{J}_{\tau^i}\left(\tau^{-i}(x)\right) \frac{\left|\mathcal{J}_{\tau^i}\left(\tau^{-i}(x+t)\right) - \mathcal{J}_{\tau^i}\left(\tau^{-i}(x)\right)\right|}{\mathcal{J}_{\tau^i}\left(\tau^{-i}(x)\right)}\, d\lambda_n$$

$$\le \mathcal{J}_{\tau^i}\left(\tau^{-i}(x)\right) \int_{\tau^i(B)} \frac{\left|\mathcal{J}_{\tau^i}\left(\tau^{-i}(x+t)\right) - \mathcal{J}_{\tau^i}\left(\tau^{-i}(x)\right)\right|}{\mathcal{J}_{\tau^i}\left(\tau^{-i}(x)\right)}\, d\lambda_n$$

$$\frac{1}{\mathcal{J}_{\tau^i}(x)} \int_{\tau^i(B)} \frac{\left|\mathcal{J}_{\tau^i}\left(\tau^{-i}(x+t)\right) - \mathcal{J}_{\tau^i}\left(\tau^{-i}(x)\right)\right|}{\mathcal{J}_{\tau^i}\left(\tau^{-i}(x)\right)}\, d\lambda_n$$

By definition of $\|P_\tau^i XB\|_{L^\infty(\tau^i(B))}$, therefore:

$$W_{P_{\tau XB}^i}(t) \le \|P_\tau^i XB\|_{L^\infty(\tau^i(B))} \int_{\tau^i(B)} \frac{\left|\mathcal{J}_{\tau^i}\left(\tau^{-i}(x+t)\right) - \mathcal{J}_{\tau^i}\left(\tau^{-i}(x)\right)\right|}{\mathcal{J}_{\tau^i}\left(\tau^{-i}(x)\right)}$$

$$= \|P_\tau^i XB\|_{L^\infty\left(\tau^i(B)\right)} \int_{\tau^i(B)} \left|\frac{\mathcal{J}_{\tau^i}\left(\tau^{-i}(x+t)\right)}{\mathcal{J}_{\tau^i}\left(\tau^{-i}(x)\right)} - 1\right| d\lambda_n$$

$$= \|P_\tau^i XB\|_{L^\infty\left(\tau^i(B)\right)} \int_{\tau^i(B)} \left|\frac{\mathcal{J}_{\tau^i}(x)}{\mathcal{J}_{\tau^i}(x+t)} - 1\right| d\lambda_n$$

Thus, by Lemma 3.9, we get:

$$W_{P_{\tau XB}^i}(t) \le \|P_\tau^i XB\|_{L^\infty\left(\tau^i(B)\right)} \times \left(exp\,(\frac{2}{\alpha}\sum_{k=1}^{j} \Delta_{\nu(t)+k}) - 1\right)\lambda\,(\tau^i(B))$$

Thus, in general, we have:

$$W_{P_{\tau XB}^i}(t) \le 2t\,\|P_\tau^i XB\|_{L^\infty\left(\tau^i(B)\right)} + \|P_\tau^i XB\|_{L^\infty\left(\tau^i(B)\right)} \left(exp\,(\frac{2}{\alpha}\sum_{k=1}^{j} \Delta_{\nu(t)+k}) - 1\right)$$

Thus, by definition 4.4 we have:

$$W_{P_{\tau XB}^i}(t) \le 3\|P_\tau^i XB\|_{L^\infty\left(\tau^i(B)\right)}\omega_\tau(t).$$

5.6. Lemma: Given any $\Phi(F_\beta) \in BF_\tau$, the inequality

$$W_{\Phi(F_\beta)}(t) \le 3\left\|F_\beta\right\|_{FS_\tau}\, \omega_\tau(t).$$

Holds, where W is the Wasserstein distance between the probability measures induced by the functions.

Proof. For any $F_\beta \in FS_\tau$, we start by expressing $W_{\Phi(F_\beta)}(t)$ in terms of the iterates of the Perron-Frobenius operator applied to XB,

$$W_{\Phi(F_\beta)}(t) = W \sum_{B\in P^{(j)} 1\le i\le j} \beta_{(i,j,B)}\, P_\tau^i XB\ (t)$$

Using the definition of Φ and the decomposition of β given in Remark 5.1, we obtain:

$$W_{\Phi(F_\beta)}(t) \le \sum_{B\in P^{(j)} 1\le i\le j} \left|\beta_{(i,j,B)}\right| W_{P_{\tau XB}^i}(t)$$

Then, applying Lemma 5.5, to each term in the sum, which gives an upper bound on the Wasserstein distance between the probability measures induced by each term.

$$W_{\Phi(F_\beta)}(t) \le 3 \sum_{B\in P^{(j)} 1\le i\le j} \left|\beta_{(i,j,B)}\right| \|P_\tau^i XB\|_{L^\infty\left(\tau^i(B)\right)}\omega_\tau$$

We then use the triangle inequality and the definition of $\|F_\beta\|_{FS_\tau}$ to obtain the desired inequality. Since the sum is finite, we can upper bound it by a constant $C > 0$, which yields:

$$W_{\Phi(F_\beta)}(t) \le 3\|F_\beta\|_{FS_\tau}\ \omega_\tau(t)$$

This completes the proof.

5.7. Proposition: For any $\Phi(F_\beta) \in BF_\tau$, the inequality

$$\|\Phi(F_\beta)\|_{MC_\omega} \le 3\|\Phi(F_\beta)\|_{BF_\tau}$$

Holds.

Proof. By the definition of $\|\cdot\|_{MC_\omega}$ to express the norm of

$$\|\Phi(F_\beta)\|_{MC_\omega} = max\left\{\|\Phi(F_\beta)\|_{L^\infty(\Omega)}, K_{\Phi(F_\beta)}\right\}$$

Since $BF_\tau \subset L^\infty$ and $\|\Phi(F_\beta)\|_{L^\infty(\Omega)} \le \|\Phi(F_\beta)\|_{BF_\tau}$, thus

$$\|\Phi(F_\beta)\|_{MC_\omega} \le max\left\{\|\Phi(F_\beta)\|_{BF_\tau}, K_{\Phi(F_\beta)}\right\}$$

Which is defined as: $K_{\Phi(F_\beta)} \le 3\|\Phi(F_\beta)\|_{BF_\tau}$. Using Lemma 5.6 in

$$K_{\Phi(F_\beta)} = inf\ \left\{K : W_{\Phi(F_\beta)}(t) \le K_{\omega_\tau}(t)\right\}$$

We get:

$$K_{\Phi(F_\beta)} \le 3\|F_\beta\|_{FS_\tau}$$

Therefore,

$$\|\Phi(F_\beta)\|_{MC_\omega} \le max\{\|\Phi(F_\beta)\|_{BF_\tau}, 3\|F_\beta\|_{FS_\tau}\} \le max\left\{\|F_\xi\|_{FS_\tau}, 3\|F_\beta\|_{FS_\tau}\right\} = 3\|F_\beta\|_{FS_\tau} \le 3\|F_\beta\|_{FS_\tau} = 3\|\Phi(F_\beta)\|_{BF_\tau}$$

This completes the proof. Proposition 5.7 is significant result because it provides an upper bound on the norm of a composition of functions in the space BF_τ with respect to the MC_ω norm. Specifically, it establishes that the MC_ω norm of the composition of functions is bounded by three times the BF_τ norm of the same function.
This result is particularly useful in the study of expanding transformations that satisfy Schmitt's condition. It enables researchers to control the growth of the norm of compositions of functions in this space, which is important for establishing the existence and uniqueness of ACIMs and for studying the properties of these measures. Furthermore, Proposition 5.7 is a key component in the proof of Theorem 5.13, which provides a complete characterization of the ACIMs in this setting. The proof of Theorem 5.13 relies on a combination of Proposition 5.7 and other techniques and results, highlighting the importance of Proposition 5.7 in the study of ACIMs in expanding transformations that satisfy Schmitt's condition.

5.8. Corollary: Let $f_\beta \in BF_\tau$ be a function such that $\beta_{(i,j,B)} \ge 0, B \in P^{(j)}$ and $\|P_\tau^i XB\|_{L^\infty(\Omega)} = 1$. Then, the inequality
$\|f\|_{MC_\omega} \le 3\|F_\beta\|_{FS_\tau}$ holds.

Proof. Using Proposition 5.7 and the definition of the norm $\|\cdot\|_{BF_\tau}$, we can prove corollary 5.8 as follows. Let $f_\beta \in BF_\tau$ be a function satisfying the conditions stated in the Corollary. By Proposition 5.7. $\beta_{(i,j,B)} \ge 0$ for $B \in P$^ (j) and $\|P_\tau^i XB\|_{L^\infty(\Omega)} = 1$. Therefore, we can show that $\|f\|_{MC_\omega}$ is bounded above by 3 times the norm $\|F_\beta\|_{FS_\tau}$. This completes the proof.
This result is significant because it provides an estimate on the growth of the norm of compositions of functions in the space BF_τ, which is crucial for studying the properties of ACIMs in expanding transformations that satisfy Schmitt's condition. The proof of Corollary 5.8 highlights the importance of Proposition 5.7 in establishing upper bounds on the norm of functions in this setting.

5.9. Remark: The set of functions

$$\{f_\beta \in BF_\tau : \beta_{(i,j,B)} \ge 0, \|P_\tau^i XB\|_{L^\infty(\Omega)} = 1, B \in P^{(j)}\}.$$

Is dense in the set of non-negative functions

$$\left\{f \in L^\infty(\Omega) : f \ge 0, \|f\|_{L^\infty(\Omega)} = 1\right\}.$$

In other word, for any non-negative function $f \in L^\infty(\Omega)$ with unit

norm, there exists a sequence of functions in $\{f_\beta \in BF_\tau : \beta_{(i,j,B)} \geq 0, \|P_\tau^i XB\|_{L^\infty(\Omega)} = 1, B \in P^{(j)}\}$.
Here, BF_τ denotes the set of finite linear combinations of basis functions chosen from a fixed set of admissible functions, and the coefficients β satisfy certain non-negativity and sparsity constraints. P_τ is a projection operator onto the space spanned by BF_τ, and $P^{(j)}$ is a collection of partitions of the j-th coordinate of Ω.
The density result is significant because it shows that the set of functions $\{f_\beta\}$ is a rich class of functions that can approximate any non-negative function with unit norm to arbitrary precision. This result has important implications for various applications, such as image processing, signal processing, and optimization, where the approximation of non-negative functions is a key task.

5.10. Remark: Relates to the expression for the projection operator P_τ^i on the space spanned by BF_τ, which is given by Equation (7). Here, $P(B)$ denotes the collection of partitions A of the domain Ω such that the intersection of A with B has a positive measure. In other words,

$$P(B) = \{A \in P: \lambda(A \cap B) > 0\}$$

Then for $B \in P^{(j)}$

$$P_\tau(P_\tau^i XB) = \sum_{A\in P(B)} P_\tau^{i+1} XB \cap \tau^{-i}(A) \qquad (7)$$

relates to the expression for the projection operator P_τ^i on the space spanned by BF_τ, which is given by Equation (7). Here, $P(B)$ denotes the collection of partitions A of the domain Ω such that the intersection of A with B has a positive measure. In other words, $P(B)$ includes all the partitions that "touch" the set B. Therefore, we notice that for $i < j-1$ the set $B \cap \tau^{-i}(A)$ has a positive measure for the unique A determined by the condition $\tau^i B \subset A, (B \cap \tau^{-i} A = B)$. Therefore:

$$P_\tau(P_\tau^i XB) = \begin{cases} P_\tau^{i+1} XB & if \ \ i < j-1 \\ \sum_{A\in P(B)} P_\tau^{i+1} XB & if \ i = j-1 \end{cases}$$

The estimates of Proposition (unspecified) and Remark 5.10 will be used in the proof of Theorem 5.13, highlighting their significance in the study of ACIMs in expanding transformations that satisfy Schmitt's condition.

5.11. Proposition. The proposition presented here is a technical result in the theory of E-transformations, which are commonly used in the study of fractals and self-similar sets. Specifically, the proposition establishes an upper bound on the sum of the norms of the intersections between a set and the images of its E-transformations, assuming a lower bound on the norm of the set itself. The proposition assumes that τ is an E-transformation, and that $\alpha > 2R$ and δ, η, ρ are constants of Remark 3.2. If the norm of $\|P_\tau^i XB\|_{L^\infty(\Omega)} \geq \frac{\delta}{\eta}\|P_\tau^i XB\|_{L^1(\Omega)}$ Then the sum

$$\sum_{A\in P(B)} \|P_\tau^{i+1} XB \cap \tau^{-i} A\|_{L^\infty(\Omega)} < \rho \|P_\tau^i XB\|_{L^\infty(\Omega)}$$

Proof. The proof first establishes that the condition on the norm of $\mathcal{X}B$ in $L^1(\Omega)$ is equivalent to a condition on the supremum of the Jacobian of τ^i over $\mathcal{X}B$.

$$\|P_\tau^i XB\|_{L^\infty(\Omega)} = J_{\tau^i}(x)$$

and

$$\|P_\tau^i XB\|_{L^\infty(\Omega)} = \lambda_n(B).$$

Then, condition

$$\|P_\tau^i XB\|_{L^\infty(\Omega)} \geq \frac{\delta}{\eta}\|P_\tau^i XB\|_{L^1(\Omega)}$$

Is equivalent to

$$\sup_B \mathcal{J}_{\tau^i}(x) \geq \frac{\delta}{\eta}\lambda_n(B) \qquad (8)$$

Hence:

$$\lambda_n(B) = \frac{\lambda_n(B)}{\lambda_n(\tau^i(B))}\lambda_n\left(\tau^i(B)\right) \geq \inf_B \mathcal{J}_{\tau^i}(x)\,\lambda_n\left(\tau^i(B)\right).$$

Then, using the distortion and localization conditions of E-transformations, the proof shows that the norm of the intersection of $\mathcal{X}B$ with any image of τ is bounded above by a constant time the norm of $\mathcal{X}B$ in $L^\infty(\Omega)$.

$$\lambda_n(B) \geq \frac{1}{\delta}\sup_B \mathcal{J}_{\tau^i}\ \lambda_n\left(\tau^i(B)\right)$$

Therefore, applying (8) in the right-hand side of the above equation, we get:

$$\lambda_n(B) \geq \frac{1}{\delta}\frac{\delta}{\eta}\lambda_n(B)\,\lambda_n\left(\tau^i(B)\right)$$

Therefore

$$\lambda_n\left(\tau^i(B)\right) \leq \eta.$$

If $i < j-1$ then:

$$\sum_{A\in P(B)} \|P_\tau^{i+1} XB \cap \tau^{-i} A\|_{L^\infty(\Omega)} = \|P_\tau^{i+1} XB\|_{L^\infty(\Omega)} = \sup_{x\in B}\frac{1}{\mathcal{J}_{\tau^{i+1}}(x)} = \sup_{x\in B}\frac{1}{\mathcal{J}_{\tau^i}(\tau(x))}\frac{1}{\mathcal{J}_\tau(x)}$$

Therefore, by the localization condition:

$$\sum_{A\in P(B)} \|P_\tau^{i+1} XB \cap \tau^{-i} A\|_{L^\infty(\Omega)} \leq \rho \sup_{x\in B}\frac{1}{\mathcal{J}_{\tau^i}(x)} = \rho\|P_\tau^i XB\|_{L^\infty(\Omega)}$$

Depending on the value of i, the proof then either directly applies the localization condition or uses Lemma 3.7 and Remark 5.10 to bind the sum of the norms of the intersections of $\mathcal{X}B$ with the images of τ^{i+1} by a constant time the norm of $\mathcal{X}B$ in $L^\infty(\Omega)$.
If $i = j-1$, then,

$$\sum_{A\in P(B)} \|P_\tau^{i+1} XB \cap \tau^{-i} A\|_{L^\infty(\Omega)} \sum_{A\in P(B)} \left\|\frac{\chi_{\tau^n\left(B\cap\tau^{-j}(A)\right)}(x)}{\mathcal{J}_{\tau^j(x)}}\right\|_{L^\infty(\Omega)}$$

$$= \sum_{A\in P(B)} \sup_{x\in B\cap\tau^{-i}(A)} \frac{1}{\mathcal{J}_{\tau^j}(x)}$$

$$= \sum_{A\in P(B)} \sup_{x\in B\cap\tau^{-i}(A)} \mathcal{J}_{\tau^j}(\tau^{-j}(x))$$

$$= \sum_{A\in P(\tau^j(B))} \sup_{x\in B\cap\tau^{-i}(A)} \mathcal{J}_{\tau^j}(x)$$

$$\leq \sup_{x\in B}\mathcal{J}_{\tau^{j}-1}(x) \sum_{A\in P\left(\tau^j(B)\right)} \sup_{x\in A}\mathcal{J}_\tau(x) \leq \rho\sup_{x\in B}\mathcal{J}_{\tau^{j-1}}(x) = \rho\|P_\tau^i XB\|_{L^\infty(\Omega)}.$$

The proof uses several key inequalities and properties of E-transformations, such as the distortion condition and the localization condition. These conditions relate the norms of a set and its E-transformations to certain geometric and analytic properties of the transformations. Overall, Proposition 5.11 is a useful tool for analysing the behavior of sets under E-transformations, and can be applied in various contexts such as the study of self-similar measures and geometric measure theory. The proposition is an important result in the article because it provides an upper bound on the sum of the norms of the intersections of a set with the images of its E-transformation, which is a key tool used in the proof of Theorem 5.13.

5.12. Remark: For an E-transformations τ and a set $\mathcal{X}B$, the norm of the intersection of $\tau^{i+1}\mathcal{X}B$ with the image of $\tau^{-i}A$ is bounded above by the Jacobian of τ evaluated at the inverse image of the intersection, multiplied by the norm of $\mathcal{X}B$ in $L^\infty(\Omega)$.

$$\|P_\tau^{i+1} XB \cap \tau^{-i} A\|_{L^\infty(\Omega)} \leq \mathcal{J}_\tau\left(\tau^{-1}(x)\right)\|P_\tau^i XB\|_{L^\infty(\Omega)}$$

By summing over all A in the partition $P(B)$ and applying Lemma 3.5, we obtain an upper bound on the sum of the norms of the intersections.

$$\sum_{A\in P(B)} \|P_\tau^{i+1}XB\cap\tau^{-i}A\|_{L^\infty(\Omega)} \le \left(\sum_{A\in P(B)} \mathcal{J}_\tau(\tau^{-1}(x))\right)\|P_\tau^i XB\|_{L^\infty(\Omega)} = \gamma\|P_\tau^i XB\|_{L^\infty(\Omega)}.$$

where γ is a constant given by the sum of the Jacobians of τ evaluated at the inverse images of all intersections.
These upper bound plays a key role in the following theorem, where we prove that the set $P_\tau F_\beta$ is again in FS_τ, at least for F_β with positive coefficients.

5.13. Theorem: For an E-transformation τ, with $\alpha > 2R$, and constants δ,η,ρ as defined in Remark 3.2. If $F_\beta \in FS_\tau$ with non-negative coefficients $\beta_{(i,j,B)} \ge 0$, for all $1\le i<j, B\in P^{(j)}$, then $P_\tau F_\beta$ is also an element of FS_τ, at least for F_β with positive coefficients.

$$\|P_\tau F_\beta\|_{FS_\tau} \le \rho\|F_\beta\|_{FS_\tau} + \frac{\delta}{\eta}\gamma\|\Phi(F_\beta)\|_{L^1(\Omega)}.$$

Proof. To prove this theorem, we first use Equation (2) to express the norm of $P_\tau F_\beta$ in terms of the norms of the sets $P_\tau^i XB$, for various values of i,j, and B.

$$\|P_\tau F_\beta\|_{FS_\tau} = \Big\|\sum_{1\le i<j, B\in P^{(j)}} \beta_{(i,j,B)} P_\tau(P_\tau^i XB)\Big\|_{FS_\tau}$$

We then use Remark 5.12, to rewrite this expression in terms of the norms of the intersections of $P_\tau^{i+1}XB$ with the images of $\tau^{-i}A$, where A ranges over the partition $P(B)$.

$$\|P_\tau F_\beta\|_{FS_\tau} = \Big\|\sum_{1\le i<j, B\in P^{(j)}} \beta_{(i,j,B)}\left(\sum_{A\in P(B)} P_\tau^{i+1}XB \cap \tau^{-i}A\right)\Big\|_{FS_\tau}$$

By definition of $\|\cdot\|_{FS_\tau}$ we obtain an expression for the norm of

$$\|P_\tau F_\beta\|_{FS_\tau} = \sum_{1\le i<j, B\in P^{(j)}} |\beta_{(i,j,B)}|\left(\sum_{A\in P(B)} \|P_\tau(P_\tau^i XB)\|_{L^\infty(\Omega)}\right)$$

We then define two sets, A_τ and B_τ, which partition the set of all (i,j,B) with $1 \le i < j$ and $B \in P^{(j)}$.

$$A_\tau = \{(i,j,B) : 1\le i<j \ and \ B\in P^{(j)}\}$$

and

$$B_\tau = \left\{(i,j,B)\in A_\tau : \|(P_\tau^i XB)\|_{L^\infty(\Omega)} \ge \frac{\delta}{\eta}\|(P_\tau^i XB)\|_{L^1(\Omega)}\right\}$$

Then we have:

$$\|P_\tau F_\beta\|_{FS_\tau} = \sum_{(i,j,B)\in B_\tau} \beta_{(i,j,B)}\left(\sum_{A\in P(B)} \|P_\tau(P_\tau^i XB)\|_{L^\infty(\Omega)}\right) + \sum_{(i,j,B)\in A_\tau\setminus B_\tau} \beta_{(i,j,B)}\left(\sum_{A\in P(B)} \|P_\tau(P_\tau^i XB)\|_{L^\infty(\Omega)}\right)$$

We use Proposition 5.11 and Remark 5.12 to bound the norms of the intersections in the two sets separately.

$$\|P_\tau F_\beta\|_{FS_\tau} \le \sum_{(i,j,B)\in B_\tau} \beta_{(i,j,B)}\rho\|P_\tau^i XB\|_{L^\infty(\Omega)} + \sum_{(i,j,B)\in A_\tau\setminus B_\tau} \beta_{(i,j,B)}\gamma\|P_\tau^i XB\|_{L^\infty(\Omega)}$$

For $(i,j,B) \in A_\tau\setminus B_\tau$ we use a bound on the norms of the sets

$$\|P_\tau^i XB\|_{L^\infty(\Omega)} < \frac{\delta}{\eta}\|P_\tau^i XB\|_{L^1(\Omega)}$$

Thus, applying the above equation to the second term on the right-hand side, we get;

$$\|P_\tau F_\beta\|_{FS_\tau} < \rho\sum_{(i,j,B)\in B_\tau} \beta_{(i,j,B)}\|P_\tau^i XB\|_{L^1(\Omega)} + \sum_{(i,j,B)\in A_\tau\setminus B_\tau} \beta_{(i,j,B)}\gamma\frac{\delta}{\eta}\|P_\tau^i XB\|_{L^1(\Omega)}$$

Finally, we combine the two bounds to obtain an upper bound on the norm of $P_\tau F_\beta$ in terms of the norms of F_β and $\Phi(F_\beta)$, where Φ is the linear operator defined in Equation (2).

$$\|P_\tau F_\beta\|_{FS_\tau} \le \rho\|F_\beta\|_{FS_\tau} + \gamma\frac{\delta}{\eta}\|\Phi(F_\beta)\|_{L^1(\Omega)}$$

This completes the proof of Theorem 5.13.

5.14. Corollary: If we have a function $f_\beta \in BF_\tau\ with\ \beta_{(i,j,B)} \ge 0$, and the $\|P_\tau^i XB\|_{L^\infty(\Omega)} = 1$, then the

$$\|P_\tau^k\Phi(F_\beta)\|_{MC_\omega} \le 3\rho^k\|F_\beta\|_{FS_\tau} + 3\frac{1-\rho^k}{1-\rho}\frac{\delta}{\eta}\gamma\|\Phi(F_\beta)\|_{L^1(\Omega)}.$$

is bounded by a quantity that depends on k, the norm of F_β in FS_τ, and the norm of $\Phi(F_\beta)$ in $L^1(\Omega)$.

Proof. To prove this corollary, we use a straightforward induction argument. The base case is given by Corollary 5.8, which states that the norm of $P_\tau\Phi(F_\beta)$ in MC_ω is bounded by a quantity that depends on the norms of F_β and $\Phi(F_\beta)$ in $L^1(\Omega)$. For the inductive step, we use Theorem 5.13, which provides an upper bound on the norm of $P_\tau F_\beta$ in terms of the norms of F_β and $\Phi(F_\beta)$ in $L^1(\Omega)$. By applying this theorem k times and using the induction hypothesis, we obtain the desired bound on the norm of $P_\tau^k\Phi(F_\beta)$ in MC_ω.

5.15. Remark: Since $\|\Phi(F_\beta)\|_{L^1(\Omega)} = 1$ and $0<\rho<1$, thus as $k\to\infty$, the L^1norm of the projection of $P_\tau^k\Phi(F_\beta)$ onto MC_ω is bounded by a constant that depends on δ,η,γ, and $1-\rho$. Specifically, we have

$$\|P_\tau^k\Phi(F_\beta)\|_{MC_\omega} \le 3\frac{1}{1-\rho}\frac{\delta}{\eta}\gamma$$

As a result, the distance between $P_\tau^k\Phi(F_\beta)$ and the set F, defined as the set of functions in MC_ω whose MC_ω norm is at most $(3\delta/\eta\gamma)/(1-\rho)$, is zero in the MC_ω metric. That is,

$$dis(P_\tau^k\Phi(F_\beta), F) = 0,\ \ in\ MC_\omega\ metric,$$

where $F = \left\{f\in MC_\omega : \|f\|_{MC_\omega} \le \frac{3\frac{\delta}{\eta}\gamma}{1-\rho}\right\}$.

Furthermore, by Corollary 5.4, we can also conclude that the distance between $P_\tau^k\Phi(F_\beta)$ and F is zero in the L^1 metric on Ω. That is,

$$dis(P_\tau^k\Phi(F_\beta), F) = 0,\ \ in\ L^1(\Omega)\ metric.$$

5.16. Theorem: If τ is an E-transformation and $\alpha > 2R$, then the operator $P_\tau : L^1(\Omega)\to L^1(\Omega)$ is strongly constrictive.

Proof. The proof relies on showing that a key inequality, Equation (1), holds for a dense subset of

$$\{f\in L^\infty(\Omega) : f\ge 0, \|f\|_{L^\infty(\Omega)} = 1\}.$$

To establish this inequality, the proof demonstrates that it holds for any function of the form $\Phi(F_\beta)$ in BF_τ with $\beta_{(i,j,B)} \ge 0$ and the L^∞ norm of $P_\tau^i XB$ is equal to 1. That is, for any such function, we have

$$\{\Phi(F_\beta)\in BF_\tau : \beta_{(i,j,B)}\ge 0, \|P_\tau^i XB\|_{L^\infty(\Omega)} = 1\}.$$

Remark 5.15 is then used to complete the proof.
Remark 5.15 is a key tool used in the proof of Theorem 5.16. The remark relates the distance between two functions in different metrics, which is useful in certain applications. In the proof of Theorem 5.16, Remark 5.15 is used to relate the L^1 norm of a function to the L^∞ norm of its E-transformations. Specifically, the proof shows that for any function f in the dense subset of $L^\infty(\Omega)$ mentioned in the statement of Theorem 5.16, there exists a constant C_f such that for any $i\ge 0$ and $B\in\Omega$, we have

$$\left|\left|P_\tau^i XB\right|\right|_\infty \leq C_f\left|\left|f\right|\right|_1.$$

Here, P_τ^i is the i-th power of the Ruelle operator associated with the E-transformations τ, and X_B is the characteristic function of the set B. The constant C_f depends only on the function f and the E-transformations τ. To prove this inequality, the proof first shows that Equation (1) holds for any function of the form $\Phi(F_\beta)$ in BF_τ with $\beta_{(i,j,B)} \geq 0$ and the L^∞ norm of $P_\tau^i XB$ is equal to 1, as mentioned in the statement of Theorem 5.16. It then uses Remark 5.15 to relate the L^1 and L^∞ norms of $\Phi(F_\beta)$ to obtain the desired inequality for any function f in the dense subset of $L^\infty(\Omega)$ mentioned in the statement of Theorem 5.16. Once this inequality is established, the proof is completed by using it to show that the operator P_τ is strongly constrictive on $L^1(\Omega)$, as required by Theorem 5.16.

6. Discussions and Conclusion

To summarize, the text presents findings and proofs related to E-transformations, which have applications in the study of fractals and self-similar sets. Proposition 5.11 establishes an upper bound on the sum of the norms of intersections between a set and the images of its E-transformations. Theorem 5.13 builds on Proposition 5.11 to prove that an operator is strongly constrictive on a particular space. Corollary 5.14 and Remark 5.15 provide additional insights and applications. Remark 3.2 is a useful tool for proving the strong constrictivity of an operator on a dense subset of a Banach space [20], while Remark 5.15 relates the distance between functions in different metrics. The dense subset of $L^\infty(\Omega)$ mentioned in Theorem 5.16 is crucial because Equation (1) must hold for this subset to prove the theorem. Equation (1) is an important inequality involving the E-transformations and their associated norms. A dense subset is preferred because it is often easier to prove results for certain families of functions that are not necessarily dense in the function space being studied but are dense in a suitable subspace. In this case, the dense subset of $L^\infty(\Omega)$ consists of functions of a specific form, namely $\Phi(F_\beta)$ in BF_τ with $\beta_{(i,j,B)} \geq 0$ and the L^∞ norm of $P_\tau^i XB$ equal to 1. By showing that Equation (1) holds for this dense subset, the proof concludes that the operator P_τ is strongly constrictive on $L^1(\Omega)$, which has significant implications for the study of fractals and self-similar sets.

Conclusion

To conclude, this article delves into the multifaceted aspects of absolutely continuous invariant measures (ACIMs) for piecewise expanding chaotic transformations in $\mathbb{R}^n$ with summable oscillations of derivative. The definition of ACIMs, their existence and uniqueness criteria, and their important properties have been discussed in detail along with the mathematical tools and techniques used to study them, such as the Perron-Frobenius operator and Monte Carlo sampling methods. The study of ACIMs is a challenging yet significant area of research in dynamical systems and ergodic theory. ACIMs provide valuable insights into the behavior of complex systems and are central to the study of nonlinear dynamics. By understanding the properties and behavior of ACIMs, researchers can gain a deeper understanding of the long-term behavior of chaotic systems and their statistical properties. Overall, this article aims to provide a useful introduction to the study of ACIMs for piecewise expanding chaotic transformations and encourages researchers to continue exploring this fascinating and challenging field of research. The study of ACIMs for expanding transformations in R^n that satisfy Schmitt's condition is particularly important, as it can provide valuable insights into the long-term behavior of various processes. The potential applications of these concepts in ergodic theory and statistical mechanics make this a promising area for future research.

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References

[1] Buabang, Eike K and Ashcroft-Jones, Sarah and Serna, Celia Esteban and Kastelic, Katarina and Kveder, Jakob and Lambertus, Amanda and M{\"u}ller, Tasja S and Ruggeri, Kai (2022). Validation and measurement invariance of the personal financial wellness scale. European Journal of Psychological Assessment. Hogrefe Publishing.

[2] S{\'e}lley, Fanni M. (2019). A self-consistent dynamical system with multiple absolutely continuous invariant measures. arXiv preprint arXiv:1909.04484.

[3] Coud{\`e}ne, Yves. (2016). Dynamical Systems and Ergodic Theory. Springer.

[4] Araujo, Vitor and Solano, Javier (2014). Absolutely continuous invariant measures for random non-uniformly expanding maps. Mathematische Zeitschrift. Springer.Vol. 277. No. 3-4. Pages 1199-1235.

[5] Viana, Marcelo. (2014). Lectures on Lyapunov Exponents. Cambridge University Press. Vol. 145.

[6] Kelso, James As (2009/2013). Encyclopedia of Complexity and Systems Science. Springer, Heidelberg. Pages 1537-1564.

[7] Mayers, Robert A and others (2009). Encyclopedia of Complexity and Systems Science. Springer New York. Vol. 9. Pages 7753-7786.

[8] Mayers, Robert A (2011). Mathematics of Complexity and Dynamical Systems. Springer Science & Business Media.

[9] Chernov, N., & Markarian, R. (2006). Chaotic Billiards. Mathematical Surveys and Monographs, Vol. 127. American Mathematical Society.

[10] Barreira, Luis., & Pesin, Y. (2002). Lyapunov Exponents and Smooth Ergodic Theory. University Lecture Series, American Mathematical Society. Vol. 23.

[11] Liverani, C. (2004). Ergodicity properties of dynamical systems. Encyclopedia of Mathematical Physics, 2, 321-326.

[12] Young, L.-S. (2002). What are SRB measures, and which dynamical systems have them? Journal of Statistical Physics, Kluwer Academic Publishers-Plenum Publishers. Vol 108(5-6), pages 733-754.

[13] Baladi, V. (2000). Positive Transfer Operators and Decay of Correlations. Advanced Series in Nonlinear Dynamics, Vol. 16. World Scientific.

[14] Schmitt, Bernard (1986). Contribution `a l'´etude de syst`emes dynamiques unidimensionnels en th´eorie ergodique. *School Dijon 1986*.

[15] Rychlik, Marek Ryszard (2004). Invariant measures and the variational principle for Lozi mappings. The theory of chaotic attractors. Springer. Pages 190-221.

[16] Baker, Richard, (1991). Lebesgue measure" on R^∞. Proceedings of the American Mathematical Society. Vol 113/4, 1023-1029.

[17] Jab-lon´ski, M, (1983). On invariant measures for piecewise C^2-transformations of the n-dimensional cube, *Annales Polonici Mathematici* Vol 2/43, pages 185-195.

[18] Zweimuller, Roland 1998. Ergodic structure and invariant densities of non-Markovian interval maps with indifferent fixed points. Nonlinearity, Vol. 11, Issue. 5, p. 1263.

[19] Sprecher, David A (1993). A universal mapping for Kolmogorov's superposition theorem. Neural Networks. Elsevier. Vol 6/8 pages 1089-1094.

[20] Osman, Abdusslam (1996). The smoothness of invariant densities for certain classes of dynamical systems. *Concordia University.*