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Generating some linear second-order differential equations with undetermined coefficients by Linearly combining Frobenius and Legendre differential equations

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Abstract This paper aims to generate some linear, second-order and homogeneous ordinary differential equations (ODEs) with variable and undetermined coefficients by linearly combining some well-known ODEs in mathematical physics community which are the Legendre and Frobenius (or Bessel as a special case) ODEs. Comparing the form of the generated ODEs with the original (mother) ODEs should allow us to facilitate choosing the appropriate method to solve such ODEs. Then solve the generated ODEs by the usual methods for solving Legendre and Frobenius ODEs which are the series or Frobenius method depending on the nature of the point that we need to solve in a neighbourhood of it. Such point should be no worse than a regular singular point. For demonstration purposes, we introduced some illustrated examples for a set of some generated linear, second-order ODEs which are solved by the aid of the mother ODEs. We claim that, this procedure grants us some easiness for solving such generated ODEs which are encountered in many physical and engineering applications.

Keywords: Legendre equation, Frobenius equation, Bessel equation, Series method, Frobenius method, Second-order linear differential equations with undetermined coefficients.

انشاء معادلات تفاضلية خطية و متجانسة من الرتبة الثانية و ذات معاملات متغيرة و غير محددة كتركيبة خطية لمعادلتي فروبينس و لجندر

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الملخص تهدف هده الورقة الى انشاء معادلات تفاضلية عادية خطية متجانسة ذات صور مختلفة من الرتبة الثانية و ذات معاملات متغيرة و غير محددة و ذلك كتركيبة خطية من المعادلتين التفاضليتين الشهيرتين (فروبينس ولجندر) و اللتين لهما نفس النوع. وبمقارنة شكل المعادلات التفاضلية المستحدثة مع شكل كلا من المعادلات الأم (فروبينس ولجندر) نتمكن من تخمين الطريقة الملائمة لحل المعادلات التفاضلية الناتجة. حيث يمكننا غالبا الاستفادة من الطرق المعتادة لحل هذا النوع من المعادلات التفاضلية المشهورة و التي عادة ماتكون باستخدام المتسلسلات أو طريقة فروبينس و ذلك حسب طبيعة النقطة المراد ايجاد الحل بجوارها . وهذا مما لا يجعل الرياضي يمكث التفكير طويلا في البحث عن الطريقة الملائمة لحل المعادلات التفاضلية التي تنتج في مختلف التطبيقات الهندسية و الفيزيائية العديدة مما يوفرالكثير من الوقت و العناء كما سنرى ذلك من خلال الأمثلة المطروحة في هذه الورقة.

الكلمات المفتاحية: معادلة لجندر التفاضلية، معادلة بسل التفاضلية، معادلة فروبينس التفاضلية، طريقة المتسلسلات، طريقة فروبينس، المعادلات التفاضلية العادية الخطية من الرتبة الثانية و ذات المعاملات المتغيرة و الغير محددة.

Introduction

Most of the mathematical models of different applications in life are governed by differential equations (DEs) that relate all the variables of the problem under consideration. To fully understand such a problem, the need emerges to solve such governing DE and understand the behaviour of the obtained solution such as convergence in the region of interest. However obtaining such solution in a closed form is not always an easy task, where sometimes one really needs to resort to some numerical techniques to seek an approximate solution. In physical and engineering applications, one usually encounters different forms of linear, second-order homogenous ODEs with variable coefficients which are close to some well-known ODEs. Thus, one needs to utilize this feature to guess the solution method to solve such equations. One of the most known powerful techniques to solve this kind of ODEs is the power-series technique [1, 2, 5, 9]. To find the solution near a certain point or points by the power series method, we first need to determine the location and nature of the singular points of the relevant ODE. The power series method with undetermined coefficients provides solutions of the linear second-order, homogenous ODEs with variable coefficients about a point which is no worse than a regular singular point. Such a method is particularly efficient when the resulting recurrence relation for the undetermined coefficients contains no more than two different subscripts of these coefficients [1, 2]. One should expect that the obtained solution by the series method satisfies the given ODE, though it may not converge over the required region. This is encountered when solving the Legendre ODE, where one really needs to stick to an appropriate choice of its index k to avoid the divergence problem as it will be explained later on through this article [1]. The success of the power series method relies on the roots of the resulting indicial equation and the degree of singularity of the variable coefficients in the given DE. These issues may affect the convergence of the obtained solution, or at worst a second solution cannot be obtained.

Fuchs's theorem [1, 2] states that the power series methods always provide at least one power-series solution about a point which can be an ordinary point or at worst a regular singular point. Thus there is no guarantee that such method will provide a second solution which we expect for a second-order ODE. For instance, the power-series method provides only one solutions for Bessel equation, thus a second linearly independent solution can be constructed for instance by means of the Wronskian double integral [1]. If we try to solve an ODE about an irregular singular point, the method may fail even in finding one solution. Fortunately most of the ODEs known in mathematical physics have no irregular singularities in the finite plane. For instance all the singular points of the Legendre [1, 2, 3, 4, 5, 7], Chebychev [4, 7, 12], Hypergeometric ODEs [1, 2, 7, 12, 15] are regular in the finite or infinite plane [1, 2, 3, 4, 5, 6, 7, 8, 9, 10]. Such ODEs are called Fuchsian equations. Whereas the Bessel, confluent hypergeometric [1, 7], Laguerre [1, 12], Hermite [1, 12] ODEs have irregular singular point only at infinity [1, 4, 5, 11, 12, 13, 14, 15, 16, 17].

Actually it should be noted that the number of singular points of a Fuchsian DE is finite and their coefficients are rational functions of the independent variable. A Fuchsian equation must have at least one singular point and at most a finite number of such points [2]. Motivated by the great importance of special functions in general and the Legendre and Frobenius ODEs in particular, here we shall generate some second-order, homogenous linear ODEs with undetermined coefficients by linearly combining the Legendre and Frobenius ODEs. This paper is constructed as follows: in section one; we will introduce some necessary definitions, such as the ordinary point and the singular point then classification of singularity. In section two, we will give a short overview on the Frobenius ODE and show how to solve such equation. In section three, we will give a short overview on the Legendre ODE and find its solution. Then in section four, we shortly introduce the Bessel ODE which is a special case of Frobenius DE. In section five, we will show how to construct some linear, second-order and homogenous ODEs with undetermined variable coefficients by linearly combining the Legendre, Bessel and Frobenius ODEs and then solve such generated ODEs. A conclusion is drawn in section six.

1. Preliminaries

Here we shall introduce some necessary concepts, which we will need later on, such as definition of the ordinary and singular points of second-order linear ODE and how to solve it by Frobenius method.

Ordinary and Singular Points of the Second-Order Linear ODEs

Definition 1 (Analytic Function): A function $f(x)$ is called analytic function at a given point x_0 if it can be expanded in Taylor series in a neighborhood of $x = x_0$, that is, $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$.

The convergence is for some radius of convergence $R > 0$. It should be noted that the analyticity of a function implies the existence of all its derivatives and also implies the continuity of this function.

Definition 2 (Ordinary Point): A linear, second-order, homogeneous ODE may be put in the form

$$y'' + P(x)y' + Q(x)y = 0. \qquad (1)$$

The point $x = x_0$ is called an ordinary point of the ODE (1) if both the functions $P(x)$ and $Q(x)$ are analytic at $x = x_0$. Another practical way to check if $x = x_0$ is an ordinary point of DE (1) is that $P(x)$and $Q(x)$ both remain finite at the point $x = x_0$.

The following theorem gives the existence of the solution of DE (1) near an ordinary point.

Theorem 1 [1, 2, 7]: If $x = x_0$ is an ordinary point of the DE (1) then there exists a unique solution of it in the neighbourhood of x_0, which is analytic, that is $y = \sum_{n=0}^{\infty} a_n(x - x_0)^n$, with initial conditions $y(x_0) = a_0$, $y'(x_0) = a_1$, and the convergence of the series is for $|x - x_0| < R, R > 0$. The general solution of the DE (1) is $y = \sum_{n=0}^{\infty} a_n(x - x_0)^n = Ay_1(x) + By_2(x)$,

where A, B are arbitrary constants and $y_1(x)$, $y_2(x)$ are linearly independent series solutions of DE (1) which are analytic at x_0. Furthermore the radius of convergence of each the series solution $y_1(x)$ and $y_2(x)$ is at least as large as the minimum of the radius of convergence of the series for $P(x)$ and $Q(x)$.

Definition 3 (Singular Point): If in the DE (1) we find that at least one of the functions $P(x)$or $Q(x)$ is not analytic at $x = x_0$, then the point $x = x_0$ is called a singular point of the DE (1).

The singular points are classified as follows:

Definition 4 (Regular Singular Point): A point $x = x_0$ in the finite plane is called a regular (nonessential) singular point of the DE (1) if $(x - x_0)P(x)$ and $(x - x_0)^2 Q(x)$ are both analytic functions at $x = x_0$.

Definition 5 (Irregular Singular Point): A point $x = x_0$ is called an irregular (essential) singular point of the DE (1) if it is not a regular singularity of DE (1) that is, if at least one of these functions

$(x-x_0)P(x)$ and $(x-x_0)^2Q(x)$ is not analytic function at $x=x_0$. The irregular singular point $x=x_0$ of the DE (1) can be also checked if at least one of these limits $\lim_{x\to x_0}(x-x_0)P(x)$ or $\lim_{x\to x_0}(x-x_0)^2Q(x)$ does not exist.

2. Frobenius Differential Equation [16]

If the DE (1) has a regular singular point at $x=x_0$ then the method used to find a series solution of (1) is valid in the neighbourhood of $x=x_0$ and it is known as the Frobenius method. We assume $y=\sum_{n=0}^{\infty}a_n(x-x_0)^{r+n}$, $a_0\neq 0$.

The point $x=x_0$ should not be chosen at an irregular singular point. It must be a regular singular point and it can be an ordinary point. Now since $x=x_0$ is a regular singular point of DE (1) then $P(x)$ or $Q(x)$ or both are not analytic at $x=x_0$ but $\lim_{x\to x_0}(x-x_0)P(x)$ and $\lim_{x\to x_0}(x-x_0)^2Q(x)$ are analytic at $x=x_0$. Consequently we can multiply the DE (1) by the factor $(x-x_0)^2$ to obtain the following ODE

$$(x-x_0)^2y''+(x-x_0)p(x)y'+q(x)y=0, \quad (2)$$

where $p(x)=(x-x_0)P(x)$, $q(x)=(x-x_0)^2Q(x)$ are analytic functions at the point $x=x_0$, that is

$$p(x)=b_0+b_1x+b_2x^2+\cdots+b_nx^n+\cdots$$
$$q(x)=c_0+c_1x+c_2x^2+\cdots+c_nx^n+\cdots$$

By choosing $x_0=0$ in the DE (2) we obtain the following ODE

$$x^2y''+xp(x)y'+q(x)y=0. \quad (3)$$

This is referred to as Frobenius ODE. The following theorem assures the existence of a Frobenius series solution of DE (2) and hence DE (1).

Theorem (Frobenius method):

If $x=x_0$ is a regular singular point of DE (1) then the functions $(x-x_0)P(x)$ and $(x-x_0)^2Q(x)$ are analytic at $x=x_0$ and have Taylor series, that is

$$(x-x_0)P(x)=\sum_{n=0}^{\infty}p_n(x-x_0)^n,$$

where, $p_0=\lim_{x\to x_0}(x-x_0)P(x)$,

$$(x-x_0)^2Q(x)=\sum_{n=0}^{\infty}q_n(x-x_0)^n,$$

where, $q_0=\lim_{x\to x_0}(x-x_0)^2Q(x)$.

Both of these series converge for $|x-x_0|<R$, where $R>0$ is the minimum of the radius of convergence of both series. If r_1 and r_2 are the real roots of the following indicial equation

$$r(r-1)+p_0r+q_0=0,$$

such that $r_1\geq r_2$ then DE (1) and hence DE (2) has a fist solution as:

$$y_1(x)=\sum_{n=0}^{\infty}a_n(x-x_0)^{n+r_1},\ a_0\neq 0.$$

The second linearly independent solution is obtained according to the following:

1. If r_1-r_2 is not an integer or zero, then

$$y_2(x)=\sum_{n=0}^{\infty}a_n(x-x_0)^{n+r_2},\ a_0\neq 0.$$

2. If $r_1=r_2$, then the second solution of DE (1) may take the form

$$y_2(x)=y_1(x)\ln|x-x_0|+(x-x_0)^{r_2}\left[1+\sum_{n=1}^{\infty}b_n(r_1)(x-x_0)^n\right]$$

3. If $r_1-r_2=N$, $N\in\mathbb{Z}^+$, then the second solution of DE (1) may take the form

$$y_2(x)=Ay_1(x)\ln|x-x_0|+(x-x_0)^{r_2}\left[1+\sum_{n=1}^{\infty}c_n(r_2)(x-x_0)^n\right]$$

.

The coefficients $a_n(r_1),a_n(r_2),b_n(r_1),c_n(r_2)$ and the constant A can be determined by substituting the series solutions for y in the DE (2). The constant A may turn out to be zero. Each series in the solutions converges at least for $|x-x_0|<R$, and define a function that is analytic at $x=x_0$ [2].

3. Legendre ODE

Consider the linear second-order ODE

$$(1-x^2)y''-2xy'+n(n+1)y=0, n=0,1,\ldots \quad (4)$$

This ODE is known as Legendre ODE and named after a French mathematician A. M. Legendre (1752-1833). Legendre DE has three regular singular points at $x=\pm1,\infty$. We should note that the coefficients functions (after dividing by the factor $(1-x^2)$ can be written as,

$$\frac{n(n+1)}{(1-x^2)}=n(n+1)(1+x^2+x^4+\cdots),$$

$$\frac{-2x}{(1-x^2)}=-2x(1+x^2+x^4+\cdots).$$

Both of these functions represented by a geometric series which are convergent in a region that is extended to the nearest singular point, that is in the region $|x|<1$. We shall solve Legendre ODE by assuming that there is a solution of the form $y=\sum_{m=0}^{\infty}a_mx^{m+r}$, $a_0\neq 0$.

Direct substitution of y into the Legendre DE yields

$$\sum_{m=0}^{\infty}(r+m)(r+m-1)a_mx^{r+m-2}-\sum_{m=2}^{\infty}(r+m-2)(r+m-3)a_{m-2}x^{r+m-2}$$

$$-2\sum_{m=2}^{\infty}(r+m-2)a_{m-2}x^{r+m-2}+n(n+1)\sum_{m=2}^{\infty}a_{m-2}x^{r+m-2}=0$$

$$\left[r(r-1)a_0x^{r-2}+r(r+1)a_1x^{r-1}\right]+x^{r-2}\left\{\sum_{m=2}^{\infty}(r+m)(r+m-1)a_mx^m\right.$$

$$\left.-\left[\sum_{m=2}^{\infty}(r+m-2)(r+m-3)a_{m-2}x^m+2\sum_{m=0}^{\infty}(r+m-2)a_{m-2}x^m-n(n+1)\sum_{m=2}^{\infty}a_{m-2}x^m\right]\right\}=0$$

By equating the coefficient of x^{r-2} to zero, one has

$$\sum_{m=2}^{\infty}\left\{(r+m)(r+m-1)a_m-\left[(r+m-2)(r+m-1)-n(n+1)\right]a_{m-2}\right\}x^m$$
$$+\left[r(r-1)a_0+r(r+1)a_1x\right]=0$$

Now we get the indicial equation $r(r-1)=0$ with roots $r_1=1, r_2=0$ and the recurrence relation

$$a_m=\frac{-\left[n-(r+m-2)\right]\left[n+(r+m-1)\right]}{(r+m)(r+m-1)}a_{m-2},$$

Let r=0 in this relation to get

$$a_m = \frac{-(n-m+2)(n+m-1)}{m(m-1)} a_{m-2}, m = 2,3,...$$

Note that we can solve Legendre ODE by general power series since x=0 is an ordinary point to get the same recurrence relation. Now from this relation we have

$$a_2 = \frac{-n(n+1)}{1\cdot 2} a_0,$$

$$a_3 = \frac{-(n-1)(n+2)}{3\cdot 2} a_1,$$

$$a_4 = \frac{(n-2)n(n+1)(n+3)}{1\cdot 2\cdot 3\cdot 4} a_0,$$

Hence,

$$a_{2k} = \frac{(-1)^k n(n-2)\cdots(n-2k+2)(n+1)\cdots(n+2k-1)}{(2k)!} a_0$$

and

$$a_{2k+1} = \frac{(-1)^k (n-1)\cdots(n-2k+1)(n+2)\cdots(n+2k)}{(2k+1)!} a_1$$

Hence the general solution of Legendre DE is

$$y(x) = a_0 y_{even} + a_1 y_{odd}$$

where

$$y_{even} = 1 + \sum_{k=0}^{\infty} \frac{(-1)^k n\cdots(n-2k+2)(n+1)\cdots(n+2k-1)}{(2k)!} x^{2k}$$

$$y_{odd} = x + \sum_{k=0}^{\infty} \frac{(-1)^k (n-1)\cdots(n-2k+1)(n+2)\cdots(n+2k)}{(2k+1)!} x^{2k+1}$$

Since we obtain a solution with two arbitrary constants, then there is no need to consider the case r = 1. We should mentions that both solutions y_{even} and y_{odd} diverge at $x = \pm 1$, but by an appropriate choice of the index n (integral values), one of the solutions turns to a polynomial. Actually, this choice has physical interpretation in quantum mechanics.

4. Bessel ODE

Consider the following linear second-order ODE,

$$x^2 y'' + xy' + (x^2 - k^2)y = 0, k = 0,1,2... \quad (5)$$

This ODE is known as Bessel DE due to a German mathematician F. Bessel (1784 −1846) and has a regular singular point at x = 0. We should note that the Bessel ODE is a special case of Frobenius ODE. Therefore it can be solved in a similar way to solving Forbenius DE where, $p(x) = 1, q(x) = x^2 - k^2$.

5. Discussion on Linearly Combining Legendre, Bessel and Frobenius ODEs

Here in this section, we show how to linearly combine some well-known DEs in mathematical physics in order to construct some DEs. In this paper we use the the most prominent ones (Legendre and Frobenius) DEs and Bessel DE which is a special case of Frobenius DE. Then we will show how to solve the generated DEs with the aid of the mother ODEs. We start with showing some examples of how to generate some ODEs. For instance, by merging and subtracting the Legendre DE (4) and Frobenius DE (3), we respectively obtain the following ODEs,

$$y'' + x[p(x) - 2]y' + [k(k+1) + q(x)]y = 0. \quad (6)$$

$$(1 - 2x^2)y'' - x[2 + p(x)]y' + [k^2 + k - q(x)]y = 0. \quad (7)$$

Also, by merging the Bessel DE (5) and Frobenius DE (3), we obtain the following DE,

$$2x^2 y'' - x[1 + p(x)]y' + [q(x) + x^2 - k^2]y = 0, \quad (8)$$

We note that using Frobenius DE in the combination process leads to some ODEs with variables coefficients which contain unknown analytic functions $P(x)$ and $Q(x)$. Thus this approach generates quite general ODEs, because the unknown functions $P(x)$ and $Q(x)$ are arbitrary analytic functions.

In all the generated DEs (6), (7), (8), we should emphasize that the index k is a nonnegative constant, and the unknown functions $p(x)$, $q(x)$ are assumed to be analytic functions, that is they inherit the same features of the original mother DEs. Now in the following demonstrations examples, we will show how to solve the generated DEs (6), (7), (8), in the neighbourhood of the point x = 0, which can be an ordinary point or at worst a regular singular point of the relevant DE. At first glance, we note that the form of the DE (7) is relatively close to the Legendre ODE, whereas the form of the DE (8) is quite close to Frobenius DE. This should considerably facilitate choosing the appropriate solution method as it will be demonstrated in the following examples.

Example 1: Solve the DE (6).

Solution: Since this DE has an ordinary point at $x = 0$, we assume the solution as $y = \sum_{n=0}^{\infty} a_n x^n$. Direct substitution of y into the DE (6) yields

$$[2a_2 + 6a_3 x + 12a_4 x^2 + \cdots]$$
$$+[p(x) - 2][a_1 x + 2a_2 x^2 + 3a_3 x^3 + \cdots] +$$
$$[q(x) + (k^2 + k)][a_0 + a_1 x + a_2 x^2 + \cdots] = 0.$$

By equating all the coefficients of x to zero, one obtains the undetermined coefficients as

$$a_2 = \frac{[q(x) + k(k+1)]}{2!} a_0,$$

$$a_3 = \frac{[2 - q(x) - p(x) - k(k+1) + 2]}{3!} a_1,$$

$$a_4 = \frac{[2p(x) + q(x) + k^2 + k - 4][q(x) + k^2 + k]}{4!} a_0,$$

So, one obtains the general solution of the DE (6) as,

$$y = a_0 \left[1 - \frac{[q(x) + (k^2 + k)]}{2!} x^2 + \cdots\right] +$$
$$a_1 \left[x + \frac{[2 - p(x) - q(x) - (k^2 + k) + 2]}{3!} x^3 + \cdots\right],$$

where a_0 and a_1 are arbitrary constants.

Example 2: Solve the ODE (7).

Solution: Since the DE (7) has an ordinary point at $x = 0$, so we assume the solution as $y = \sum_{n=0}^{\infty} a_n x^n$. Direct substitution of y into the DE (7) yields

$$(1 - x^2)[2a_2 + 6a_3 x + 12a_4 x^2 + \cdots] -$$

$[2+p(x)][a_1x+2a_2x^2+3a_3x^3+\cdots]+$
$[k^2+k-q(x)+][a_0+a_1x+a_2x^2+\cdots]=0.$

By equating all the coefficients of x to zero, one obtains the undetermined coefficients as

$$a_2=\frac{[q(x)-k(k+1)]}{2!}a_0,$$
$$a_3=\frac{[q(x)+p(x)-k(k+1)+2]}{3!}a_1,$$
$$a_4=\frac{[2p(x)+q(x)-(k^2+k)][q(x)-(k^2+k)]}{4!}a_0,$$

So, one obtains the general solution of the ODE (7) as,

$$y=a_0\left\{1+\frac{[q(x)-k(k+1)]}{2!}x^2+\cdots\right\}+$$
$$a_1\left\{x+\frac{p(x)+[q(x)-k(k+1)+2]}{3!}+\cdots\right\}$$

Example 3: Solve the DE (8).

Solution: Since the DE (8) possesses a regular singular point at $x=0$, so we assume that there is a solution of the form $y=\sum_{n=0}^{\infty}a_nx^{m+n}$, $a_0\neq 0$. Also, we have

$p(x)=b_0+b_1x+b_2x^2+\cdots+b_nx^n+\cdots,$
$q(x)=c_0+c_1x+c_2x^2+\cdots+c_nx^n+\cdots$

Direct substitution of $p(x)$ and $q(x)$ into the DE (8) yields

$$\left[2x^2\begin{pmatrix}a_0m(m-1)x^{m-2}+a_1(m+1)mx^{m-1}+\\a_0(m+2)(m+1)x^m+\ldots+\\a_n(m+n)(m+n-1)+\ldots\end{pmatrix}-\right.$$
$$-x\left(b_0+b_1x+b_2x^2+\ldots\right)\begin{pmatrix}a_0mx^{m-1}+a_1(m+1)x^m+\\a_2(m+2)x^{m+1}+\\+\ldots+a_n(m+n)x^{m+n-1}+\ldots\end{pmatrix}+$$
$$\left.+\begin{matrix}\left(-k^2+c_0+c_1x+(1+c_2)x^2+c_3x^3+\ldots\right)\\\left(a_0x^m+a_1x^{m+1}+a_2x^{m+2}+\ldots+a_nx^{m+n}\right)\end{matrix}\right]=0$$

which can be written as

$$\left\{a_0\left[2m(m-1)-m(1+b_0)-k^2+c_0\right]\right\}x^m+$$
$$+\begin{Bmatrix}a_1\left[2m(m+1)-(m+1)(1+b_0)-k^2+c_0\right]+\\a_0\left[c_1-b_1m\right]\end{Bmatrix}x^{m+1}+\ldots+$$
$$\begin{Bmatrix}a_n\left[2(m+n)(m+n-1)-(m+n)(1+b_0)-k^2+c_0\right]+\\a_{n-1}\left[c_1-b_1(m+n-1)\right]+a_{n-2}\left[(1+c_2)-b_2(m+n-2)\right]+\\\ldots+a_0\left[c_n-b_nm\right]\end{Bmatrix}x^{m+n}+$$
$$+\ldots=0$$

By equating the coefficient of x^m to zero, we obtain the indicial equation. Then by equating the coefficient of x^{m+n} to zero, we obtain the recurrence equation which should give the nth term of the undetermined coefficient, that is

$a_n[2(m+n)(m+n-1)-(m+n)(1+b_0)+c_0-k^2]$
$+a_{n-1}[c_1-b_1(m+n-1)]+$
$a_{n-2}[(1+c_2)-b_2(m+n-2)+\cdots+a_0[c_n-b_nm]]=0.$

6. Conclusion

Here we showed that combining Frobenius ODE linearly with other ODEs of the same kind generates a set of DEs of the same kind. Then we solved these generated ODEs in the neighbourhood of the point x = 0, which can be an ordinary point or at worst a regular singular point, though the solution about a non-zero point can be obtained in a similar way. By substituting the unknown analytic functions p(x) and q(x), one can easily obtain the general solution of the relevant DE as demonstrated by the examples presented above. It should be noted that comparing the form of the generated ODEs with the mother ODEs gives us a first glance of how to solve such DEs as shown by the examples demonstrated above. This considerably facilitates and saves great time in searching the appropriate method to solve such ODEs. As future work, this approach can be extended to other well-known ODEs, such as Hermite, Laguerre, Hypergeometric ODEs, etc.

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