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Fifth Order Improved Runge-Kutta Method for Random Initial Value Problems

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Abstract When the initial conditions of differential equations are random in natural, mathematical description of the solution in terms of initial conditions must be modified by considering the initial conditions as random variable having a particular statistical distribution. In this paper, the fifth order improved Runge-Kutta method (IRK5) is modified for the approximation of the solution of the random initial value problems (RIVPs). Numerical simulation was carried out. Some properties of the random behaviour and the effect of the randomness were investigated.

Keywords: Fifth order improved Runge-Kutta method, Random initial value problems, Triangular distribution.

إستحدام طريقة رنج-كوتا المطورة من الرتبة الخامسة لإيجاد الحل العددي لمسائل القيمة الابتدائية العشوائية

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الملخص عندما تكون الشروط الابتدائية للمعادلات التفاضلية وبالاخص لمسائل القيمة الابتدائية قيماً عشوائية، فأن التعبير الرياضي الذي يصف الحل يجب ان يعدل. هذا التعديل يجب ان يأخد في الأعتبار هذه الشروط الابتدائية هي متغيرات عشوائية تملك توزيعات احصائية معينة. في هذه الورقة، عدلت طرقة رنج–كوتا المطورة من الدرجة الخامسة لايجاد الحل العددي العشوائي لمسائل القيمة الابتدائية العشوائية. المحاكاة العددية انجزت في هذه الورقة وأيضاً نوقشت بعض الخواص الاحصائية للحل العشوائي. وتأثيرالعشوائية على الحل أيضاً.

الكلمات المفتاحية: طريقة رنج كوتا المطورة من الدرجة الخامسة، مسئلة القيمة الابتدائية العشوائية، التوزيع الاحصائي المثلثي.

Introduction

Most real word phenomena is described by differential equations. In our life, we have learned to accept we are actually dealt with uncertainty which in particular can arise when the initial states of the modelling that describing the phenomena might require an educated guessed value or are not clearly known. Most realistic mathematical models that are described by the initial value problems with random initial conditions cannot be analytically solved. In the case, the analytical solution is not available; instead, they must be dealt by computational methods that deliver approximate solutions. Numerical solution of random differential equations has been addressed by Cortés et al. [1-3] and Calbo et al. in [4]. Those authors deal with the numerical solution construction of random initial value problems, which means a random Euler method. The methods are based on a sample treatment and on conditions expressed in terms of the mean square behaviour of the right-hand side of RIVPs. This approach has the advantage that conclusions remain true in a deterministic case. Jentzen and Neuenkirch [5] studied a random Euler scheme for the approximation of Carathéodory differential equations. It was reported that under weak assumptions, this approximation scheme obtains the same rate of convergence as the classical Monte–Carlo method for integration problems. Jentzen and Kloeden [6] used Pathwise Taylor schemes for solving random ordinary differential equations. A class of random boundary value problems was studied by Barry and Boyce [7]. They dealt with the nonlinear two point boundary value problem by considering some parametric as bounded continuous random variables. Here, modified IRK5 is picked up for the approximation of the solution of the RIVPs. Numerical simulation is carried out. Some properties of the random behaviour and the effect of the randomness is investigated.

Improved Runge-Kutta Method

The initial value problem has the form:

$$\begin{cases} x'(t) = F\big(t, x(t)\big), \\ \quad x(t_0) = x_0 \end{cases} \tag{1}$$

where $F: \mathbb{R}_+ \times E$ and E is an open subset of $\mathbb{R}^n$. The general form of the IRK method is [8]:

$$x_{n+1} = (1-\delta)x_n - \delta x_{n-1} + h\left(b_1 k_1 - b_{-1} k_{-1} + \sum_{i=1}^{s} b_i (k_i - k_{-i})\right) \tag{2}$$

where $n = 1,2,\dots,N-1$. With δ as a constant and the constant h as a step-size of the iterations. The values b_{-1}, b_1 are the order conditions of IRK method of order five and a_{ij} are non-zero constants for every i and j. The constants k_{-i} and k_i can be evaluated by

$$\begin{cases} k_1 = F(t_n, x_n) \\ k_{-1} = F(t_{n-1}, x_{n-1}) \\ k_i = F\left(t_n + c_i h, x_n + h\sum_{j=1}^{i-1} a_{ij}k_j\right) \\ k_{-i} = F\left(t_{n-1} + c_i h, x_{n-1} + h\sum_{j=1}^{i-1} a_{ij}k_{-j}\right) \end{cases} \quad (3)$$

for $i = 2,3,\dots,s$, $c_2,\dots,c_s \in [0,1]$. Without loss the generality, the fifth order IRK method can be derived with $\delta = 0$ as follows

$$x_{n+1} = x_n + h\left(b_1k_1 - b_{-1}k_{-1} + \sum_{i=1}^{s} b_i(k_i - k_{-i})\right) \quad (4)$$

where $n = 1,2,\dots,N-1$ and

$$\begin{cases} k_1 = F(t_n, x_n) \\ k_{-1} = F(t_{n-1}, x_{n-1}) \\ k_i = F\left(t_n + c_i h, x_n + h\sum_{j=1}^{i-1} a_{ij}k_j\right) \\ k_{-i} = F\left(t_{n-1} + c_i h, x_{n-1} + h\sum_{j=1}^{i-1} a_{ij}k_{-j}\right) \end{cases} \quad (5)$$

For $i = 2,\dots,5$, $c_i \in [0,1]$, and $c_i = \sum_{j=1}^{i-1} a_{ij}$, $i = 2,\dots,5$, (See Table 1).

Table 1: General Coefficients of IRK5 method

0					
c_2	a_{21}				
c_3	a_{31}	a_{32}			
c_4	a_{41}	a_{42}	a_{43}		
c_5	a_{51}	a_{52}	a_{53}	a_{54}	
b_{-1}	b_1	b_2	b_3	b_4	b_5

The coefficients of IRK5 method in Table 1 was determined in [8] as in Table 2.

Table 2: Coefficients of IRK5 method

0					
0.25	0.25				
0.25	−0.0082	0.2586			
0.5	0.3860	0.5312	0.6444		
0.75	0.2060	−0.9002	0.8918	0.5517	
0.0222	1.0222	0.0403	−0.1070	−0.1	0.6444

The number of function evaluation is lower and almost two-steps in nature, and is computationally more efficient compared with the classical RK5 method.

The Modified Method

Consider the random initial value problem:

$$\begin{cases} x'(t) = F\big(t, x(t)\big), \\ x(t_0) = X_0 \end{cases} \quad (6)$$

where X_0 is a random variable has a known probability distribution with density $Pr(X_0)$. Here, the technique of selecting a random sample from a particular distribution was used and the IRK5 method was modified to solve these kinds of problem. (See Figure 1).

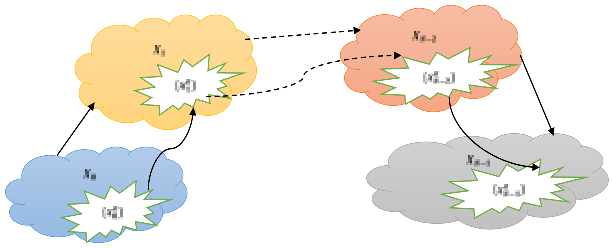

Fig. 1: The process of obtaining the random solution using a sample drawing from the population.

Now, assuming that the set $S_0 = \{x_0^\alpha : \alpha \in I, I \subset \mathbb{N}\}$ is a random sample drawn from the population of X_0. In fact, the reactions of those included in the sample should agree reasonably well (but not necessarily exactly) with the population. However, it can be saying this because the basic concepts of probability allow us to make an inference and draw conclusions about the characteristics of the interested population from which the sample was drawn. Therefore, the IRK5 method that defined in equations (4) and (5) can be modified as:

$$x_{n+1}^{\alpha} = x_n^{\alpha} + h(b_1k_1^{\alpha}b_{-1}k_{-1}^{\alpha} + \sum_{i=1}^{s} b_i(k_i^{\alpha} - k_{-i}^{\alpha})) \quad ; \alpha \in I \quad (7)$$

where $n = 1,2,\dots,N-1$, and

$$\begin{cases} k_1^{\alpha} = F(t_n, x_n^{\alpha}) \\ k_{-1}^{\alpha} = F(t_{n-1}, x_{n-1}^{\alpha}) \\ k_i^{\alpha} = F\left(t_n + c_i h, x_n^{\alpha} + h\sum_{j=1}^{i-1} a_{ij}k_j^{\alpha}\right) \\ k_{-i}^{\alpha} = F\left(t_{n-1} + c_i h, x_{n-1}^{\alpha} + h\sum_{j=1}^{i-1} a_{ij}k_{-j}^{\alpha}\right) \end{cases} \quad (8)$$

for $i = 2,\dots,5$, $c_i \in [0,1]$, $c_i = \sum_{j=1}^{i-1} a_{ij}$, $i = 2,\dots,5$, and $\alpha \in I$. The parameters in equations (7) and (8) are defined in Table 1 and Table 2. The set S_ζ represents a sample of the population of the random approximation at any iteration t_ζ, $\zeta = 1,2,\dots,N-1$ (See Figure 1).

$$\Pr(x) = \begin{cases} 0, & x < a \\ \dfrac{2(x-a)}{(b-a)(c-a)} & a \le x \le c \\ \dfrac{2(b-x)}{(b-a)(b-c)} & c \le x \le b \\ 0, & x > b \end{cases} \quad (9)$$

The cumulative distribution function is given by

$$CDF = \begin{cases} \dfrac{(x-a)^2}{(c-b)(b-a)} & a \le x \le c \\ 1 - \dfrac{(b-x)^2}{(b-c)(b-a)} & c < x \le b \\ 1 & x > b \end{cases} \quad (10)$$

The Modified Method

In the following numerical simulation, nonlinear random initial value problem is used to investigate the effects of considering the initial condition distributed as a triangular distribution.

Consider the initial value problem:

$$\begin{cases} x'(t) = \dfrac{x\cos(t+x)}{2}, & t \in [0,20] \\ x(t_0) = X_0 \end{cases} \tag{11}$$

where X_0 is a random variable distributed as triangular distribution with minimum value 0.5, maximum value 1.5 and mode 1. The algorithms in equations (7) and (8) are applied to solve the RIVP (11). The random solutions is approximated as shown in Figure 2. The behavior of the probability density function of the initial state with known statistical properties will affect the behavior of the probability density function of the random solution at any time.

The determination of this family of distribution functions constitutes the solution that will be obtained numerically since the problem (11) has no closed form solution.

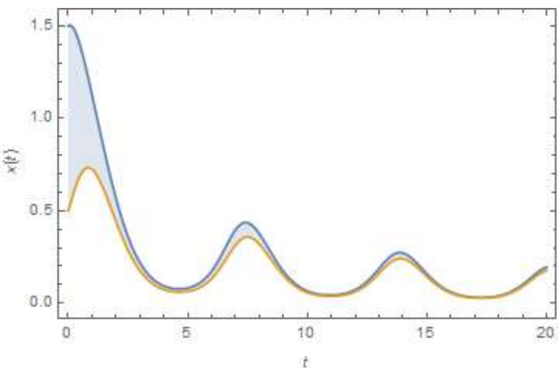

Fig. 2: The random behavior of the RIVP over time

Note that, for every specific time t, the PDFs can be numerically obtained which generally describes the distribution, relative likelihood and the probabilistic behavior of the solution. Figure. 3 illustrates the PDFs of the random solution at different iterations which is initially symmetric around the mean and takes different shapes over time.

The cumulative distribution function is another useful way to describe the distribution of the random solution. It can be calculated via the probability distribution function which is helpful to characterize the probability measurement underlying the random variable by integrating the probability density function; that is, it gives the probability that the variable will have a value less than or equal to any selected value. In the current study, the cumulative distribution functions of the solution is numerically obtained where its curves allow us to infer whether the distribution has low or high degree of kurtosis which will accordingly give us more information about the random solution. The movement of the cumulative distribution function of the solution over time is illustrated in Figure 4.

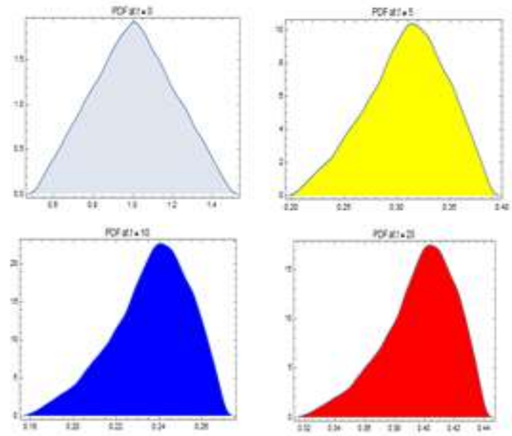

Fig. 3*:* The PDFs of the random solution at different time.

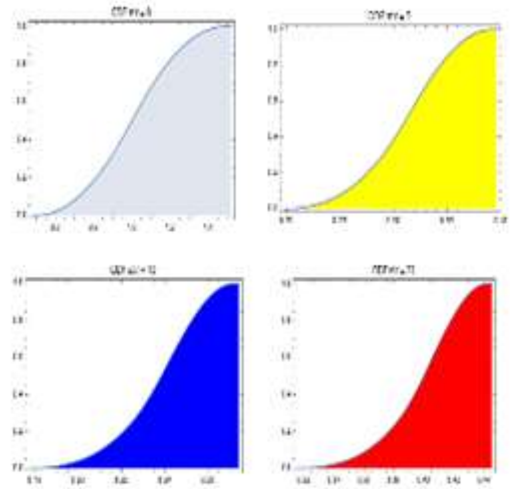

Fig. 4: The CDFs of the random solution at different time.

In addition, determining the probability density function of the random solution enables us to determine a number of the statistical properties associated with the solution, such as the moments of the solution. The moments especially the mean and the variance are surely among the most important features associated with the random solution.

Figure 5, 6 and 7 shows the mean, the variance and the range of the random solution. It can be seen that the mean curve follows the crisp solution at the initial condition $x_0 = 1$. The behavior of the variance takes its maximum at $t = 0$ and decreases over time. The range, which calculates the distance between the minimum and the maximum values over time, is also has its maximum at the beginning and becomes smaller over time. That means whichever value you pick in that interval is going to be close to the true value.

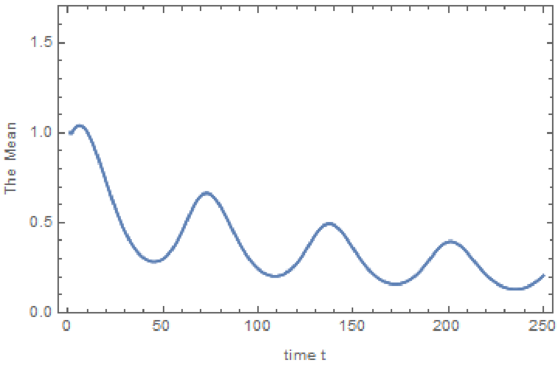

Fig. 5: The mean of the random solution over time

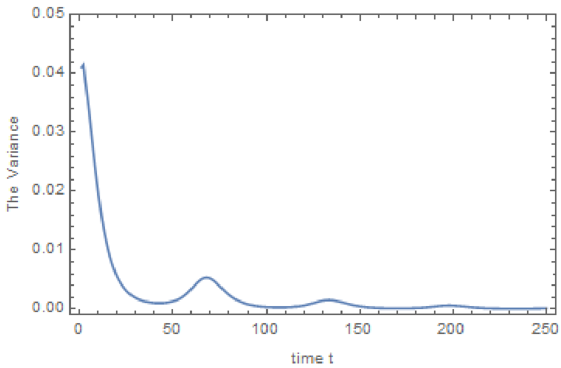

Fig. 6: The variance of the random solution over time

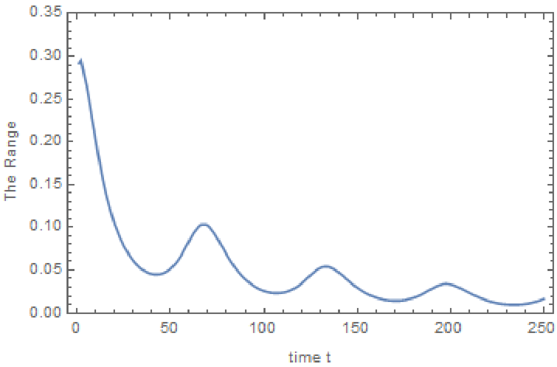

Fig. 7: The range of the random solution over time

Conclusion

The modified Improved Runge-Kutta method of order five for solving random initial value problems was developed in this paper. The technique of drawing sample from known statistical distribution to compute the random solution was incorporated. The triangular distribution is chosen to be the initial condition of the differential equation. Numerical simulation is carried out and some properties about the moments is marked. Moreover, the solution of the differential equation will not be classical; it will take the form of randomness that is limited by the range, which is different over time. In general, the results showed that, although the randomness is introduced in one area of the differential equation, the crisp behavior of the solution are randomly affected.

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