On Shehu Transform with Application of Solutions of Fractional Differential Equations

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Abstract

This review article explains the Shehu transform as a tool used for solving linear differential equations of fractional order, where the definition of the Caputo differential operator of order \( \alpha > 0 \), is taken into consideration. The transformation is used to convert Initial Value Problems (IVPs) of the fractional order of Caputo sense into simple algebraic equations. Then the inverse of the transform is used to obtain the analytical solution of the problem. We solved some illustrative examples.

Keywords: Shehu transform, Fractional integral, Fractional derivative, Caputo operator, IVP of fractional order

Preliminaries

In this section, we introduce some basics that need to be well known for proceeding in solving fractional differential equations.

Definition 1.

The function is defined as,

\[ E_{\alpha}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\alpha + 1)}, \quad \alpha \in \mathbb{C}, \quad Re(\alpha) > 0, \]

is called the Mittag-Leffler function. Further generalization of Mittag-Leffler function is introduced as

\[ E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\alpha + \beta)}, \quad \alpha, \beta \in \mathbb{C}, \quad Re(\alpha) > 0, \quad Re(\beta) > 0. \]

Definition 2.

If \( v(t) \), \( t > 0 \) is a real function, then it said to be in the space \( C_{\mu}^n \), if there exists a real number \( p > \mu \) so that \( v(t) = t^p h(t) \), where \( h(t) \in C([0, \infty)), \) and it is said to be in the space \( C_{\mu}^n \), if \( v^{(n)} \in C_{\mu}^n \).

Definition 3.

The fractional derivative of \( f(t) \) in the Caputo sense is defined as follows,

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\[ cD^\alpha v(t) = \frac{1}{\Gamma(n - \alpha)} \int_0^t (t - \xi)^{n-\alpha-1} v^{(n)}(\xi) d\xi, \quad n-\alpha > 0, \quad n \in \mathbb{N} \]

where \( n-\alpha > 0 \), \( n \in \mathbb{N} \), \( \alpha \in (0, 1) \).

Recently, Shehu Maitama Shehu [10] introduced a new integral transform. The transform is called Shehu transform, which is used in solving ordinary and partial differential equations [3].

**Definition 4.** [10].

Shehu transform of the function \( v(t) \) of exponential order is defined over the set of functions,

\[ A = \left\{ v(t) : \exists N, \eta_1, \eta_2 > 0, |v(t)| < N \exp \left( \frac{t}{\eta_1} \right), \text{if } t \in (-1)^j \times [0, \infty) \right\} \]

by the following integral

\[ \mathcal{S}[v(t)] = V(s, u) = \int_0^\infty \exp \left( \frac{-st}{u} \right) v(t) dt, \quad t > 0 \]

or

\[ v(t) = \mathcal{S}^{-1}[V(s, u)] = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{1}{u} \exp \left( \frac{st}{u} \right) V(s, u) ds, \quad t \geq 0, \]

where \( s \) and \( u \) are the Shehu transform variables, and \( \alpha \) is a real constant, the integral in Eq.(4) is taken along \( s = \alpha \) in the complex plane \( s = x + iy \).

**Theorem 1.** (The sufficient condition for the existence of Shehu transform). If the function \( v(t) \) is piecewise continuous in every finite interval \( 0 \leq t \leq \beta \), and of exponential order \( \alpha \) for \( t > \beta \), then its Shehu transform \( V(s, u) \) exists. Proof: see [10].

**Properties of Shehu Transform:**

1. Shehu transform is a linear operator. If the functions \( \alpha v(t) \) and \( \beta w(t) \) be in the set \( A \), then \( (\alpha v(t) + \beta w(t)) \in A \), where \( \alpha \) and \( \beta \) are nonzero arbitrary constants, and \( \mathcal{S}[(\alpha v(t) + \beta w(t))] = \alpha \mathcal{S}[v(t)] + \beta \mathcal{S}[w(t)]. \)

Proof: Trivial

2. Change of scale property of Shehu transform. Let the function \( v(bt) \) be in the set \( A \), where \( b \) is an arbitrary constant. Then

\[ \mathcal{S}[v(bt)] = \frac{u}{b} V \left( \frac{s}{b} \right) u. \]

Proof: see [10].

3. If \( v^{(n)}(t) \) is the \( n \)th derivative of the function \( v(t) \) in \( A \) with respect to \( t \), then its Shehu transform is given by the formula

\[ \mathcal{S}[v^{(n)}(t)] = \frac{u^n}{n!} V(s, u) - \sum_{k=0}^{n-1} \sum_{k=0}^{n-1} \frac{u^n}{k!} V^{(k)}(0). \]

4. Suppose \( V(s, u) \) and \( W(s, u) \) are the Shehu transforms of \( v(t) \) and \( w(t) \), respectively, both defined in the set \( A \). Then the Shehu transform of the convolution of two functions is defined by

\[ \mathcal{S}[(v * w)(t)] = V(s, u) W(s, u), \]

where the convolution of two functions is defined by \( (v * w)(t) = \int_0^t v(\xi)w(t - \xi) d\xi = \int_0^t v(t - \xi)w(\xi) d\xi. \)

5. Some special Shehu transforms are

\[ \mathcal{S}[t^n] = \left( \frac{u^n}{n!} \right) \Gamma(n + 1), \quad n = 0, 1, 2, \ldots \]

**Theorem 2.** Let \( n \in \mathbb{N}^* \) and \( \alpha > 0 \) such that \( n - 1 < \alpha < n \) and \( V(s, u) \) the Shehu transform of the function \( v(t) \). Then the Shehu transform denoted by \( V^\alpha_C(s, u) \) of the Caputo fractional derivative of \( v(t) \) of order \( \alpha \) is given by

\[ \mathcal{S} \left( cD^\alpha v(t) \right) = V^\alpha_C(s, u) = \frac{S^\alpha}{u^\alpha} V(s, u) - \sum_{k=0}^{n-1} \left( \frac{s}{u} \right)^{\alpha - k - 1} D^k v(t) \bigg|_{t=0}^\infty. \]

**Theorem 3.** If \( \alpha, \beta > 0 \) and \( |\alpha| < s^{-\alpha} \), then

\[ \mathcal{S}^{-1} \left( \frac{u^\alpha s^{\alpha-\beta}}{s^\alpha + \alpha u^\alpha} \right) = t^{\beta-1}E_{\alpha, \beta}(-at^\alpha). \]

Proof: See [12].

**Discussion**

In this section, we apply the Shehu transform into the Caputo sense, within the use of the Shehu integral transform technique and its inverse, we solve two examples of IVPs of fractional order. This method is considered an additional tool added to other transformations that are used to find solutions to differential equations of fractional order.

**Example 1.** Consider the linear fractional initial value problem given as

\[ cD^\alpha v(t) + v(t) = 0, \quad v(0) = 1. \]

From Eq. (9), we obtain

\[ \mathcal{S} \left( cD^\alpha v(t) \right) = \left( \frac{s}{u} \right)^\alpha V(s, u) - \left( \frac{s}{u} \right)^{\alpha - 1}. \]

Applying the Shehu transform on both sides of Eq.(10) yields \( (\frac{s}{u})^\alpha V(s, u) - \left( \frac{s}{u} \right)^{\alpha - 1} + V(s, u) = 0 \),

\[ V(s, u) = \left( \frac{s}{u} \right)^{\alpha - 1} \quad \text{and} \quad V(s, u) = \left( \frac{s}{u} \right)^{\alpha - 1}. \]

Applying the Shehu inverse on both sides of Eq. (11) we obtain \( v(t) = E_{\alpha, \beta}(-at^\alpha) \).

Some solution plots for several values of \( \alpha \) are shown in Figure 1.

**Example 2.** Consider the initial value problem (Bagley-Torvik equation)
\[ D^\alpha v(t) + D^{3/2} v(t) + v(t) = t + 1, \quad v(0) = v'(0) = 1. \] (12)

From the relation (7) we have
\[ \mathcal{S}[v'(t)] = \frac{s^2}{u^2} V(s, u) - \frac{s}{u} v(0) - v'(0) = \frac{s^2}{u^2} V(s, u) - \frac{s}{u} - 1, \]

from Eq. (9), we obtain
\[ \mathcal{S}[D^{3/2} v(t)] = \frac{s^{3/2}}{u^{3/2}} V(s, u) - \left(\frac{s}{u}\right)^{1/2} v(0) - \left(\frac{s}{u}\right)^{1/2} v'(0) \]
\[ = \frac{s^{3/2}}{u^{3/2}} V(s, u) - \left(\frac{s}{u}\right)^{1/2} - \left(\frac{s}{u}\right)^{1/2}. \]

also,
\[ \mathcal{S}[v(t)] = V(s, u), \quad \mathcal{S}[t + 1] = \frac{u^2}{s^2} + \frac{u}{s}. \]

Applying the Shehu transformation on both sides of Eq.(12) one ends to
\[ \left(\frac{s^2}{u^2} V(s, u) - \frac{s}{u} - 1 + \left(\frac{s}{u}\right)^{3/2} V(s, u) - \left(\frac{s}{u}\right)^{1/2} - \left(\frac{s}{u}\right)^{1/2}\right) = \left(\frac{u^2}{s^2} + \frac{u}{s}\right). \]

Then solving for \( V(s, u) \), we obtain
\[ \left(\frac{s^2}{u^2} + \left(\frac{s}{u}\right)^{3/2} + 1\right) U(s, u) = \left(\frac{u^2}{s^2} + \frac{u}{s} + \frac{s}{u} + 1 + \left(\frac{s}{u}\right)^{3/2} + \left(\frac{s}{u}\right)^{1/2}\right) \]
\[ \left(\frac{s^2}{u^2} + \left(\frac{s}{u}\right)^{3/2} + 1\right)V(s, u) = \left(\frac{u^2}{s^2} + \frac{u}{s} + \frac{s}{u} + 1 + \left(\frac{s}{u}\right)^{3/2} + \left(\frac{s}{u}\right)^{1/2}\right) \]
\[ V(s, u) = \frac{u^2}{s^2} + \frac{u}{s}. \]

By applying the Shehu inverse transform we get
\[ \mathcal{S}^{-1}[V(s, u)] = \mathcal{S}^{-1}\left[\frac{u^2}{s^2} + \frac{u}{s}\right], \]

which yields
\[ v(t) = 1 + t. \]

**Conclusion**

This review article introduced the Shehu integral transform and some of its properties. We found the transform is a useful tool to solve initial value problems involving fractional order differentiations using the Caputo definition. The transformation is applied to convert initial value problems of fractional order into a simpler algebraic equation that can be easily solved then an analytic solution is obtained by using the inverse of the Shehu transform. With the use of the Shehu integral transform technique, the obtained results in terms of elementary mathematical functions agree well with those achieved using other integral transforms such as Laplace Transform. Hence, the Shehu transform is considered as an additional tool for solving continuous dynamical systems of fractional order of Caputo type and it is closely connected with the Sumudu transform.

**Abbreviations and Acronyms**

IVBs: Initial Value Problems.

**References**


