SEBHA UNIVERSITY JOURNAL OF PURE & APPLIED SCIENCES VOL.22 NO. 3 2023
DOI: 10.51984/JOPAS.V22I3.2731

جامعة سبها للعلوم البحتة والتطبيقية مجلة
Sebha University Journal of Pure & Applied Sciences

Journal homepage: www.sebhau.edu.ly/journal/index.php/jopas

On Shehu Transform with Application of Solutions of Fractional Differential Equations

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Keywords:
Shehu transform
Fractional integral
Fractional derivative
Caputo operator
IVP of fractional order

ABSTRACT

This review article explains the Shehu transform as a tool used for solving linear differential equations of fractional order, where the definition of the Caputo differential operator of order α>0, is taken into consideration. The transformation is used to convert Initial Value Problems (IVPs) of the fractional order of Caputo sense into simple algebraic equations. Then the inverse of the transform is used to obtain the analytical solution of the problem. We solved some illustrative examples.

تحويل شهو وتطبيقه في حلول المعادلات التفاضلية الكسرية

فتحي بريبش

قسم الرياضيات، كلية العلوم، جامعة الزاوية، الزاوية 023، ليبيا

الكلمات المفتاحية:
تحويل شهو
التكامل الكسري
المشتقة الكسرية
عامل كابوتو
IVP من الرتيبة الكسرية

الملخص

نستعرض بمقالة المراجعة هذه تحويل شهو كأداة مستخدمة لحل المعادلات التفاضلية الخطية ذات الرتبة الكسرية ، مع الأخذ في الإعتبارتعريف المؤثر التفاضلي لكبوتو من الرتبة α>0. يتم استخدام تحويل شهو لتحويل مسائل القيمة الإبتدائية (IVPs) ذات الرتبة الكسرية لتفاضل كبوتو إلى معادلات جبرية بسيطة ومن ثم حلها، ثم يتم استخدام معكوس التحويل للحصول على الحل التحليلي للمسألة وقد عرضنا بالدراسة حل بعض الأمثلة التوضيحية.

Introduction

TheIntegral transformations are successfully used as effective tools for solving differential and integral equations [1]-, [2]-. Fractional Differential Equations (FDEs) are an important topic due to their application in modelling problems in many fields including mechanics, electroanalytical chemistry, electrical circuits, and other physical, chemical, biological, and economic aspects [3]-, [4]-, [5]-, [6]-, [7]-. Due to their importance, researchers investigated many ideas searching for solutions to FDEs. Some of these ideas involve integral transformations such as Laplace and Sumudu transforms [8]-, [9]- that are used to solve differential equations of fractional order in the form of initial value problems (IVPs). Recently, the Shehu transform has been derived from the classical Fourier integral transform as a generalization of Laplace and Sumudu transforms [10]-. It is successfully used to deal with fractional derivatives in the Caputo sense [11]- that can be used to model various natural phenomena involving fractional derivatives.

This work is considered as a review study of what was stated in the literature. We focus on the definition and some properties of the Shehu transform for derivatives in the Caputo sense. Shehu transform is applied to IVPs of Caputo derivatives of fractional order to convert them into an algebraic equation. We used the transformation to obtain analytical solutions of fractional models as IVPs of fractional order by applying the inverse fractional Shehu transform. The study supplied illustrative examples.

Preliminaries

In this section, we introduce some basics that need to be well known for proceeding in solving fractional differential equations.

Definition 1.

The function is defined as,

$$E_\alpha(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(n\alpha+1)},\qquad \alpha\in\mathbb{C},\qquad Re(\alpha)>0,$$

is called the Mittag-Leffler function. Further generalization of Mittag-Leffler function is introduced as

$$E_{\alpha,\beta}(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(n\alpha+\beta)},\qquad \alpha,\beta\in\mathbb{C},\qquad Re(\alpha)>0,\qquad Re(\beta)>0.$$

Definition 2.

If $v(t),\ t>0$ is a real function, then it said to be in the space $C_\mu, \mu\in\mathbb{R}$ if there exists a real number $p>\mu$ so that $v(t)=t^p h(t)$, where $h(t)\in C([0,\infty))$, and it is said to be in the space C_μ^n, if $v^{(n)}\in C_\mu$, $n\in\mathbb{N}$

Definition 3.

The fractional derivative of $f(t)$ in the Caputo sense is defined as follows,

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Article History : Received 24 June 2023 - Received in revised form 22 September 2023 - Accepted 02 October 2023

$$ {}^{c}D^{\alpha} v(t) = I^{n-\alpha} D^{n} v(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\xi)^{n-\alpha-1} v^{(n)}(\quad (3) $$

where $n-1<\alpha\le n,\ n\in\mathbb{N},\ v\in C_{-1}^{n}$.

Recently, Shehu Maitama Shehu [10]- introduced a new integral transform. The transform is called Shehu transform, which is used in solving ordinary and partial differential equations [3]-.

Definition 4. [10]-.

Shehu transform of the function $v(t)$ of exponential order is defined over the set of functions,

$$ A = \left\{ v(t) \colon \exists\, N, \eta_1, \eta_2 > 0, |v(t)| < N \exp\left(\frac{|t|}{\eta_i}\right), if\ t \in (-1)^i \times [0,\infty) \right\}, $$

by the following integral

$$ \mathbb{S}[v(t)] = V(s,u) = \int_0^{\infty} \exp\left(\frac{-st}{u}\right) v(t)dt, \qquad t>0 $$

$$ = \lim_{\alpha\to\infty} \int_0^{\alpha} \exp\left(\frac{-st}{u}\right) v(t)dt, \quad s>0, \qquad u>0. $$

It converges if the limit of the integral exists, and diverges otherwise. The Shehu inverse transform is given by

$$ \mathbb{S}^{-1}[V(s,u)] = v(t), \quad t\ge 0\,, $$

or

$$ v(t) = \mathbb{S}^{-1}[V(s,u)] $$

$$ = \frac{1}{2\pi i} \int_{\alpha-i\infty}^{\alpha+i\infty} \frac{1}{u} \exp\left(\frac{st}{u}\right) V(s,u)\, ds, \qquad (4) $$

where s and u are the Shehu transform variables, and α is a real constant, the integral in Eq.(4) is taken along $s=\alpha$ in the complex plane $s = x+iy$.

Theorem 1. (The sufficient condition for the existence of Shehu transform). If the function $v(t)$ is piecewise continuous in every finite interval $0\le t\le\beta$, and of exponential order α for $t>\beta$, then its Shehu transform $V(s,u)$ exists.

Proof: see [10]-.

Properties of Shehu transform:

1- Shehu transform is a linear operator. If the functions $\alpha v(t)$ and $\beta w(t)$ be in the set A, then $(\alpha v(t)+\beta w(t)) \in A$, where α and β are nonzero arbitrary constants, and

$$ \mathbb{S}[\alpha v(t)+\beta w(t)] = \alpha\mathbb{S}[v(t)] + \beta\mathbb{S}[w(t)]. \qquad (5) $$

Proof: Trivial

2- Change of scale property of Shehu transform. Let the function $v(\beta t)$ be in the set A, where β is an arbitrary constant. Then

$$ \mathbb{S}[v(\beta t)] = \frac{u}{\beta} V\left(\frac{s}{\beta}, u\right). \qquad (6) $$

Proof: see [10]-.

3- If $v^{(n)}(t)$ is the n^{th} derivative of the function $v(t)\in A$ with respect to t, then its Shehu transform is given by the formula

$$ \mathbb{S}\left[v^{(n)}(t)\right] = \frac{s^n}{u^n} V(s,u) - \sum_{k=0}^{n-1} \left(\frac{s}{u}\right)^{n-(k+1)} v^{(k)}(0). \qquad (7) $$

4- Suppose $V(s,u)$ and $W(s,u)$ are the Shehu transforms of $v(t)$ and $w(t)$, respectively, both defined in the set A. Then the Shehu transform of their convolution is given by

$$ \mathbb{S}[(v * w)(t)] = V(s,u)\, W(s,u), \qquad (8) $$

where the convolution of two functions is defined by

$$ (v * w)(t) = \int_0^t v(\xi) w(t-\xi) d\xi = \int_0^t v(t-\xi) w(\xi) d\xi. $$

5- Some special Shehu transforms are

$$ \mathbb{S}(t^{\alpha}) = \left(\frac{u}{s}\right)\Gamma(\alpha+1). $$

$$ \mathbb{S}\left(\frac{t^n}{n!}\right) = \left(\frac{u}{s}\right)^{n+1}, \qquad n = 0,1,2,\dots $$

Theorem 2. Let $n\in\mathbb{N}^*$ and $\alpha>0$ such that $n-1<\alpha<n$ and $V(s,u)$ the Shehu transform of the function $v(t)$. Then the Shehu transform denoted by $V_{\alpha}^{c}(s,u)$ of the Caputo fractional derivative of $v(t)$ of order α is given by

$$ \mathbb{S}\left({}^{c}D^{\alpha} v(t)\right) = V_{\alpha}^{c}(s,u) = \frac{s^{\alpha}}{u^{\alpha}} V(s,u) - \sum_{k=0}^{n-1} \left(\frac{s}{u}\right)^{\alpha-(k+1)} D^k v(t)|_{t=0}. \qquad (9) $$

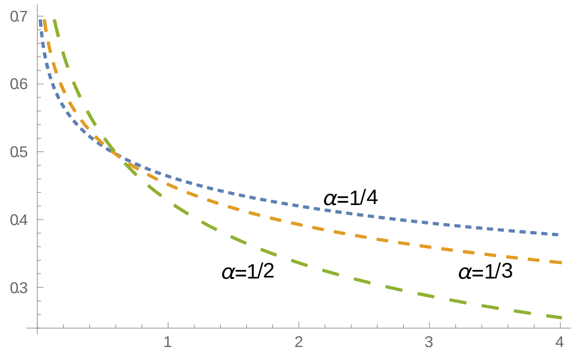

Fig. 1: Solutions $v(t)$ of the linear fractional initial value problem given by Eq. (10), where t is in the interval (0, 4], for different values, $\alpha = \frac{1}{4}, \frac{1}{3}, \frac{1}{2}$.

Proof: See [12]-.

Theorem 3. If $\alpha,\beta>0,\ \alpha\in\mathbb{R}$ and $|\alpha| < \frac{s^{\alpha}}{u^{\alpha}}$, then

$$ \mathbb{S}^{-1}\left(\frac{u^{\beta} s^{\alpha-\beta}}{s^{\alpha}+a u^{\alpha}}\right) = t^{\beta-1} E_{\alpha,\beta}(-a t^{\alpha}). $$

Proof: See [12]-.

Discussion

In this section, we apply the Shehu transformation and its properties to obtain the analytic solution of linear differential equations as IVPs. We consider differentiations of order $\alpha>0$ that are defined in the Caputo sense, within the use of the Shehu integral transform technique and its inverse, we solve two examples of IVPs of fractional order. This method is considered an additional tool added to other transformations that are used to find solutions to differential equations of fractional order.

Example 1. Consider the linear fractional initial value problem given as

$$ {}^{c}D^{\alpha} v(t) + v(t) = 0, \qquad v(0) = 1. \qquad (10) $$

From Eq. (9), we obtain

$$ \mathbb{S}[\, {}^{c}D^{\alpha} v(t)] = \left(\frac{s}{u}\right)^{\alpha} V(s,u) - \left(\frac{s}{u}\right)^{\alpha-1}. $$

Applying the Shehu transform on both sides of Eq.(10) yields

$$ \left(\frac{s}{u}\right)^{\alpha} V(s,u) - \left(\frac{s}{u}\right)^{\alpha-1} + V(s,u) = 0, $$

$$ \left(\left(\frac{s}{u}\right)^{\alpha} + 1\right) V(s,u) = \left(\frac{s}{u}\right)^{\alpha-1}, $$

$$ V(s,u) = \left(\frac{s}{u}\right)^{\alpha-1} \left(\frac{u^{\alpha}}{s^{\alpha}+u^{\alpha}}\right) = \frac{u\, s^{\alpha-1}}{s^{\alpha}+u^{\alpha}}. \qquad (11) $$

Applying the Shehu inverse on both sides of Eq. (11) we obtain

$$ v(t) = E_{\alpha}(-t^{\alpha}). $$

Some solution plots for several values α are shown in Figure 1.

Example 2. Consider the initial value problem (Bagley-Torvik equation)

$$D^2v(t) + {}^cD^{3/2}v(t) + v(t) = t + 1, \quad v(0) = v'(0) = 1. \tag{12}$$

From the relation (7) we have

$$\mathbb{S}[v''(t)] = \frac{s^2}{u^2}V(s,u) - \frac{s}{u}v(0) - v'(0) = \frac{s^2}{u^2}V(s,u) - \frac{s}{u} - 1,$$

from Eq. (9), we obtain

$$\mathbb{S}\left[{}^cD^{3/2}v(t)\right] = \frac{s^{3/2}}{u^{3/2}}V(s,u) - \left(\frac{s}{u}\right)^{\frac{1}{2}}v(0) - \left(\frac{s}{u}\right)^{-\frac{1}{2}}v'(0)$$

$$= \frac{s^{\frac{3}{2}}}{u^{\frac{3}{2}}}V(s,u) - \left(\frac{s}{u}\right)^{\frac{1}{2}} - \left(\frac{s}{u}\right)^{-\frac{1}{2}}.$$

also,

$$\mathbb{S}[v(t)] = V(s,u),$$

$$\mathbb{S}[t+1] = \frac{u^2}{s^2} + \frac{u}{s}.$$

Applying the Shehu transformation on both sides of Eq.(12) one ends to

$$\left(\frac{s}{u}\right)^2 V(s,u) - \frac{s}{u} - 1 + \left(\frac{s}{u}\right)^{3/2} V(s,u) - \left(\frac{s}{u}\right)^{\frac{1}{2}} - \left(\frac{s}{u}\right)^{-\frac{1}{2}} + V(s,u) = \left(\frac{u}{s}\right)^2 + \frac{u}{s}$$

Then solving for $V(s,u)$, we obtain

$$\left(\left(\frac{s}{u}\right)^2 + \left(\frac{s}{u}\right)^{3/2} + 1\right)V(s,u) = \left(\frac{u}{s}\right)^2 + \frac{u}{s} + \frac{s}{u} + 1 + \left(\frac{s}{u}\right)^{\frac{1}{2}} + \left(\frac{s}{u}\right)^{-\frac{1}{2}}$$

$$\left(\left(\frac{s}{u}\right)^2 + \left(\frac{s}{u}\right)^{3/2} + 1\right)V(s,u) = \left(\left(\frac{u}{s}\right)^2 + \frac{u}{s}\right)\left(\left(\frac{s}{u}\right)^2 + 1 + \left(\frac{s}{u}\right)^{\frac{3}{2}}\right)$$

$$V(s,u) = \frac{u^2}{s^2} + \frac{u}{s}.$$

By applying the Shehu inverse transform we get

$$\mathbb{S}^{-1}[V(s,u)] = \mathbb{S}^{-1}\left[\frac{u^2}{s^2} + \frac{u}{s}\right],$$

which yields

$$v(t) = 1 + t.$$

Conclusion

This review article introduced the Shehu integral transform and some of its properties. We found the transform is a useful tool to solve initial value problems involving fractional order differentiations using the Caputo definition. The transformation is applied to convert initial value problems of fractional order into a simpler algebraic equation that can be easily solved then an analytic solution is obtained by using the inverse of the Shehu transform. With the use of the Shehu integral transform technique, the obtained results in terms of elementary mathematical functions agree well with those achieved using other integral transforms such as Laplace Transform. Hence, the Shehu transform is considered as an additional tool for solving continuous dynamical systems of fractional order of Caputo type and it is closely connected with the Sumudu transform.

Abbreviations and Acronyms

IVBs: Initial Value Problems.

References

[1]- Aggarwal, Sudhanshu, Anjana Rani Gupta, and Swarg Deep Sharma. "A new application of Shehu transform for handling Volterra integral equations of first kind." *International Journal of Research in Advent Technology* 7.4 (2019): 439-445.

[2]- Aggarwal, Sudhanshu, and Anjana Rani Gupta. "Shehu transform for solving Abel's integral equation." *Journal of Emerging Technologies and Innovative Research* 6.5 (2019): 101-110.

[3]- Khan, Hassan, et al. "Analytical solutions of (2+ time fractional order) dimensional physical models, using modified decomposition method." Applied Sciences 10.1 (2020): 122.

[4]- Patil, Dinkar. "Application of integral transform (Laplace and Shehu) in chemical sciences." *DP Patil, Application Of Integral Transform (Laplace And Shehu) In Chemical Sciences, Aayushi International Interdisciplinary Research Journal, Special* 88 (2022).

[5]- Shah, R., A. Saad Alshehry, and W. Weera. "A Semi-Analytical Method to Investigate Fractional-Order Gas Dynamics Equations by Shehu Transform. Symmetry 2022, 14, 1458." (2022).

[6]- Maitama, Shehu, and Weidong Zhao. "Homotopy perturbation Shehu transform method for solving fractional models arising in applied sciences." *Journal of Applied Mathematics and Computational Mechanics* 20.1 (2021): 71-82.

[7]- Akinyemi, Lanre, and Olaniyi S. Iyiola. "Exact and approximate solutions of time-fractional models arising from physics via Shehu transform." *Mathematical Methods in the Applied Sciences* 43.12 (2020): 7442-7464.

[8]- Fahad, Hafiz Muhammad, Mujeeb Ur Rehman, and Arran Fernandez. "On Laplace transforms with respect to functions and their applications to fractional differential equations." *Mathematical Methods in the Applied Sciences* 46.7 (2023): 8304-8323

[9]- Al-Zhour, Zeyad, et al. "New results on the conformable fractional Sumudu transform: theories and applications." *International Journal of Analysis and Applications* 17.6 (2019): 1019-1033.

[10]- Maitama, Shehu, and Weidong Zhao. "New integral transform: Shehu transform a generalization of Sumudu and Laplace transform for solving differential equations." *arXiv preprint arXiv:1904.11370* (2019).

[11]- Caputo, Michele, and Mauro Fabrizio. "A new definition of fractional derivative without singular kernel." *Progr. Fract. Differ. Appl* 1.2 (2015): 1-13.

[12]- Khalouta, Ali, and Abdelouahab Kadem. "A New Method to Solve Fractional Differential Equations: Inverse Fractional Shehu Transform Method." Applications & Applied Mathematics 14.2 (2019).