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A Numerical Study for Uncertainty in two Predators-One Prey Model

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ABSTRACT

In this article, a mathematical model had been presented, which delved into the dynamics of a system featuring two predators and one prey, characterized by a growth rate of the predator dependent on a specific ratio. When facing uncertain initial conditions in the model, it had become imperative to adapt the mathematical descriptions by treating these initial conditions as random variables governed by specific distribution functions. The beta distribution had been employed to represent the stochastic initial state of the two predators-one prey model, allowing for an examination of how the random initial state of the prey had impacted the dynamic behavior of the system. Given the lack of an analytical solution for the model, computer simulations had been resorted to in order to delve into the dynamics of the system. Consequently, various properties of the resulting numerical solution had been thoroughly discussed, shedding light on the intricacies of the system's behavior. In addition, the research had offered invaluable insights into understanding the impact of uncertainty on predator-prey interactions and the subsequent stability and behavior of the ecological system in uncertain conditions, further bolstered by the numerical results obtained.

دراسة عددية لعدم اليقين في نموذج رياضي لفريسة ومُفترسَيْن

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الكلمات المفتاحية:
توزيع بيتا
الشروط الأولية العشوائية
طريقة رنج-كوتا
نموذج ثنائي المفترس- فريسة
عدم اليقين

الملخص

في هذه الورقة، تم دراسة نموذج رياضي بنظام يتضمن اثنين من الفرائس وفريسة واحدة، وتتميز بمعدل نمو يعتمد على نسبة معينة. عندما تكون القيم الابتدائية للمجتمعات غير مؤكدة في النموذج، أصبح من الضروري تكييف الوصف الرياضي من خلال معالجة هذه الشروط الابتدائية على أنها متغيرات عشوائية تخضع لدالة توزيع احصائية محددة. من اجل ذلك، تم استخدام التوزيع الاحصائي بيتا لتمثيل الحالة الابتدائية العشوائية للنموذج، مما يسمح بدراسة كيفية تأثير الحالة الابتدائية العشوائية للفريسة على السلوك الديناميكي للنظام. نظرًا لعدم وجود حل تحليلي للنموذج، تم اللجوء إلى المحاكاة العددية على الحاسوب لاستكشاف تأثيرها على ديناميكيات النظام. ونتيجة لذلك، تمت مناقشة مختلف خصائص الحل العددي الناتج، بتسليط الضوء على تفاصيل سلوك النظام. حيث قدم البحث رؤى واضحة لفهم تأثير عدم اليقين على تفاعلات المفترسين والفريسة وثبات النظام وسلوكه في ظروف غير مؤكدة، مع تعزيز ذلك بالنتائج العددية المحصلة.

Introduction

In the realm of ecology and population dynamics, predator-prey interactions play a pivotal role in shaping the stability and diversity of ecosystems. To better understand these intricate relationships, mathematical models have been developed to describe the dynamics between predator and prey populations. However, real-world ecosystems are often characterized by inherent uncertainties, making it challenging to predict the precise outcomes of these interactions. Numerous mathematical modelling scenarios inherently encompass a degree of uncertainty. This inherent uncertainty stems from the distinctions between a mathematical model and reality, arising from our limited knowledge about specific components within the model and the precise values of its parameters. Moreover, the assumptions made during the modelling process are often an approximation of reality, rather than an absolute truth. Consequently, embracing and accounting for these uncertainties becomes crucial for developing more accurate and reliable mathematical models that better reflect the

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complexities of the real-world phenomena they seek to represent [1]. Employing differential equations represents a rational approach to model dynamic systems amid the presence of possibility uncertainty. Differential equations serve as valuable tools to examine how populations evolve over time. Within this framework, various forms of uncertainty can be effectively considered. Specifically, when the initial conditions of the system are subject to randomness, differential equations offer a powerful means to account for and analyze the implications of such uncertainties. By integrating uncertainty into the mathematical representation, these models provide deeper insights into the behaviour and responses of dynamic systems in the face of unpredictable factors, leading to a more comprehensive understanding of real-world phenomena [2]. In 2010, Pollett et al. introduced a comprehensive approach to integrate random initial conditions into population models. Their methodology involved the derivation of large-scale models, encompassing the total variation arising from both the initial conditions and random fluctuations. By considering this amalgamation of factors, the researchers achieved a more robust representation of population dynamics, enabling a deeper understanding of the system's behavior under the influence of uncertainty [3]. A novel mathematical model depicting the temporal interactions among two predators and one prey was introduced. Within this model, the existence conditions of equilibrium points and their stability were derived for the three-dimensional system. By exploring these equilibrium points and their stability, the study contributes valuable insights into the dynamics and stability of the predator-prey interactions, shedding light on the intricate relationships between the species involved in the ecological system [4]. The concept of uncertainty has long been recognized in the domain of statistics. Researchers have come to acknowledge that the majority of real-world phenomena and physical experiments cannot be entirely described or predicted due to the inherent lack of knowledge about certain components within the models. This awareness of uncertainty has led to the development of various methodologies and statistical techniques to account for and quantify the impact of unknown factors, ultimately enhancing our ability to make more informed and robust conclusions in the face of uncertainty [5]. In the study conducted by Omar and Hasan, the impact of uncertain initial sizes in a predator-prey model was thoroughly investigated [2]. To address this uncertainty, the researchers employed the powerful tools of probability theory, which allowed them to quantitatively analyze and understand the potential variations in the model's outcomes. Additionally, they utilized the concepts and methodologies of fuzzy set theory [6,7] to further enhance their understanding of the complex interactions between predators and prey under uncertain conditions. By combining these analytical approaches, the study offers valuable insights into the dynamics and behaviour of the predator-prey system, accounting for the inherent uncertainties that exist in real-world ecological scenarios.

This paper focuses on investigating uncertainty in more complex predator-prey models using random theories. It aims to propose more accurate numerical methods to reduce uncertainty in simulations and precisely describe model behaviors. Uncertainty in biological models may require radical changes in traditional analysis approaches, leading to the acceptance of probabilistic or random descriptions of biological model behavior. The main objective is to introduce uncertainty in the two predator-one prey model through initial conditions, assuming the Beta distribution as prey's initial states. The objectives include modifying a numerical mathematical model to solve the uncertain model and studying the model behavior under non-crisp parameter values or initial conditions, analyzing the impact of randomness on prey and predator behavior.

1. Preliminary concepts

2.1. Two Predators-One Prey Model

Let $\boldsymbol{x(t)}$ be the density of prey species and $\boldsymbol{y_i(t)(i=1,2)}$ be the density of predator species that compete each other for the prey [8]. The Holling type-II functional response is used to describe feeding of the two predators $\boldsymbol{y_1(t)}$ and $\boldsymbol{y_2(t)}$ on prey $\boldsymbol{x(t)}$. Then the dynamics of the system may be governed by the following system of differential equations:

$$\begin{cases} \dfrac{dx(\boldsymbol{t})}{dt} = rx(\boldsymbol{t})\left(1-\dfrac{x(\boldsymbol{t})}{K}\right) - \dfrac{a_1x(\boldsymbol{t})y_1(\boldsymbol{t})}{1+b_1x+y_1+my_2} - \dfrac{a_2mx(\boldsymbol{t})y_2(\boldsymbol{t})}{1+b_2x+y_1+my_2}, \\ \dfrac{dy_1(\boldsymbol{t})}{dt} = -\delta_1y_1(\boldsymbol{t}) - \alpha y_1(\boldsymbol{t})y_2(\boldsymbol{t}) + \dfrac{\lambda_1a_1x(\boldsymbol{t})y_1(\boldsymbol{t})}{1+b_1x+y_1+my_2}, \\ \dfrac{dy_2(\boldsymbol{t})}{dt} = -\delta_2y_2(\boldsymbol{t}) - \beta y_1(\boldsymbol{t})y_2(\boldsymbol{t}) + \dfrac{\lambda_2a_2mx(\boldsymbol{t})y_2(\boldsymbol{t})}{1+b_2x(\boldsymbol{t})+y_1(\boldsymbol{t})+my_2(\boldsymbol{t})}, \\ \qquad x(0)>0, \qquad y_i(0)>0, \qquad i=1,2. \end{cases} \tag{1}$$

In the context of model **1**, several key parameters are involved., r represents the intrinsic growth rate of the prey species, while K denotes the carrying capacity of the environment. For the predator species, the parameter δ_i corresponds to their mortality rate coefficients, and α, β signify the inter-specific interference coefficients. Moreover, a_1, a_2 represent constants that determine the searching efficiency of the predators, and m represents the relative predation rate of $y_2(t)$ with respect to $y_1(t)$. λ_i stands for the food conversion coefficient specific to the predator species $y_i(t)$, $\frac{a_1}{b_1}$ and $\frac{a_2m}{b_2}$ are the maximum percapita capturing rates for $y_1(t), y_2(t)$ respectively [4]. To proceed with the analysis, the variables in model 1 will be re-scaled. Let us now delve into the implications of these re-scaled variables.

$$\begin{cases} \dfrac{dx(t)}{dt} = rx(t)\big(1-x(t)\big) - \dfrac{a_1x(t)y_1(t)}{1+b_1x(t)+y_1(t)+y_2(t)} \\ \qquad - \dfrac{a_2xy_2}{1+b_2x+y_1+y_2} = x(t)L(x(t),y_1(t),y_2(t)), \\ \dfrac{dy_1(t)}{dt} = -\delta_1y_1(t) - \alpha y_1(t)y_2(t) + \dfrac{\lambda_1a_1x(t)y_1(t)}{1+b_1x(t)+y_1(t)+y_2(t)} \\ = y_1M_1(x(t),y_1(t),y_2(t)), \\ \dfrac{dy_2(t)}{dt} = -\delta_2y_2(t) - \beta y_1(t)y_2(t) + \dfrac{\lambda_2a_2mx(t)y_2(t)}{1+b_2x(t)+y_1(t)+my_2(t)} \\ \qquad = y_2(t)M_2(x(t),y_1(t),y_2(t)), \end{cases} \tag{2}$$

All solutions of $(x(t), y_1(t), y_2(t)))$ of system (2) with the initial condition are positive for all $t>0$.

2.2. Equilibrium Points

The equilibrium points of a two-predator-one-prey mathematical model are fundamental states where the population dynamics of predators and prey reach a stable configuration the system (2) has five nonnegative equilibria [4,8], namely

$$E_0(0,0,0), E_1(1,0,0), E_2(\tilde{x},\tilde{y}_1,0), E_3(\hat{x},0,\hat{y}_2),\ E_4(\bar{x},\bar{y}_1,\bar{y}_2).$$

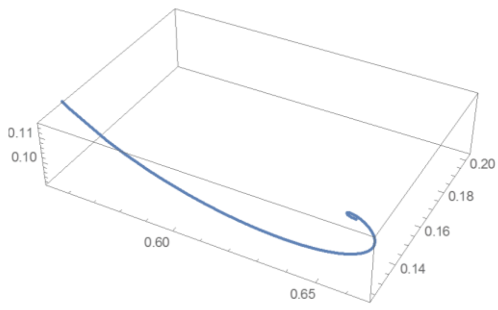

Fig. 1: The interaction of three populations over time:

In such models, the equilibrium points represent scenarios where the rates of prey consumption and predator population growth balance out, resulting in a steady coexistence or extinction of the species involved. Analyzing these equilibrium points provides valuable insights into the long-term behaviour of the ecological system, shedding light on the stability and interactions between the predators and prey. Understanding the conditions for the existence and stability of these equilibrium points is crucial for predicting the dynamics of real-world predator-prey interactions and exploring potential conservation strategies in complex ecosystems.

2.3. Beta Distribution $[\alpha, \beta]$

The Beta distribution [α,β] is a fundamental probability distribution widely used in various fields, such as statistics, engineering, and biology. It is characterized by two shape parameters, α and β, which determine the distribution's shape and behaviour. The Beta distribution is bounded on the interval [0, 1], making it particularly useful for modelling random variables that are constrained within this range. It is versatile and can assume a wide range of shapes, including symmetric, skewed, and U-shaped distributions, depending on the values of α and β. The distribution finds practical applications in modelling proportions, probabilities, and continuous data that are limited between zero and one. Due to its flexibility and ability to handle various scenarios, the Beta distribution serves as a powerful tool for uncertainty modelling, particularly in situations where uncertain quantities, such as initial conditions in mathematical models, are best represented by continuous probabilities within the range interval $[0, 1]$. [9]

2. Methodology

The methodology employed in the discussed context focuses on modelling predator-prey interactions under uncertainty, particularly using a stochastic approach. The initial conditions of the population processes are acknowledged to be uncertain, and to capture this uncertainty, the beta distribution is adopted as the assumed probability distribution for the prey's initial condition. By incorporating the beta distribution into the system of differential equations, the model accounts for random variations in the initial population sizes and allows for a more comprehensive analysis of the dynamics. This stochastic approach enables a deeper exploration of the system's behaviour under different probabilistic scenarios, shedding light on the impact of uncertainty on the stability and interactions between predators and prey. Overall, this methodology serves as a valuable tool to study the effects of uncertainty in complex ecological systems and contributes to a better understanding of predator-prey dynamics in the face of varying initial conditions.

3.1. Two Predators-One Prey Model with the Random Initial States

In the context of system (**2**), the initial conditions of the population processes may be provided; however, they cannot be precisely known due to inherent uncertainty. To address this uncertainty, the beta distribution is adopted as the assumed probability distribution for the initial condition of the prey. As a result, the system of differential equations becomes stochastic in nature, reflecting the random nature of the initial conditions and enabling a more comprehensive analysis of the population dynamics under uncertain scenarios

$$
\begin{cases}
\dfrac{dX(t)}{dt} = X(t)\big(1 - X(t)\big) - \dfrac{\rho X(t) Y_1(t)}{1 + h_1 \rho X} \\
\qquad - \dfrac{\mu X(t) Y_2(t)}{1 + h_2 \mu Y}, \quad X(0) = X_0, \\
\dfrac{dY_1(t)}{dt} = -u Y_1(t) - c_1 Y_1(t) Y_2(t) + \dfrac{e_1 \rho X(t) Y_1(t)}{1 + h_1 \rho X(t)} \\
- \dfrac{e_1 \rho}{\big(1 + h_1 \rho X(t)\big)} Y_1(t)^2, \ \ Y_1(0) = y_{1_0}, \\
\dfrac{dY_2(t)}{dt} = -w Y_2(t) - c_2 Y_1(t) Y_2(t) + \dfrac{e_2 \mu X(t) Y_2(t)}{1 + h_2 \mu X(t)} \\
\quad - \dfrac{e_2 \mu}{\big(1 + h_2 \mu X(t)\big)} Y_2(t)^2, \qquad Y_2(0) = y_{2_0}.
\end{cases} \tag{3}
$$

Where $X_0 \sim \text{Beta} - \text{Distribution}\ [\alpha_0, \beta_0]$. The value α_0 is considered the crisp initial state of the prey x_0 in the system (**3**). This system is nonlinear. In that case, a numerical method is needed to solve the random system. It will be done by modifying the 4th-order Runge-Kutta method.

3.2. The 4th order Runge Kutta for the system of two predators-one prey model

In the modified 4th order Runge-Kutta method, the differential equations are approximated using a step-by-step process. At each step, the method calculates intermediate values of the dependent variables based on the rates of change provided by the differential equations. It then takes a weighted average of these intermediate values to update the dependent variables at the next time step. This weighted averaging helps improve the accuracy of the numerical solution, making the method more robust and suitable for systems with complex dynamics. To do that, the set $\{X_0^s : s = 1, \dots, m\}$ **is** assumed to be a random sample distributed as $Beta\ [\alpha_0, \beta_0]$, then the modified equations of 4th order Range-Kutta Method are defined as

$$X_{i+1}^s = X_i^s + (K_1^s + 2K_2^s + 2K_3^s + K_4^s)\, h/6,$$

$$Y_{1_{i+1}}^s = {Y_{1_i}}^s + (L_1^s + 2L_2^s + 2L_3^s + L_4^s)\, h/6,$$

$$Y_{2_{i+1}}^s = {Y_{2_i}}^s + (J_1^s + 2J_2^s + 2J_3^s + J_4^s)\, h/6.$$

where $i = 1,2,3, \dots, n$, and

$$
\begin{cases}
K_1^s = F_1\big(t_i, {X_i}^s, {Y_{1_i}}^s, {Y_{2_i}}^s\big) \\
L_1^s = F_2\big(t_i, {X_i}^s, {Y_{1_i}}^s, {Y_{2_i}}^s\big) \\
J_1^s = F_3\big(t_i, {X_i}^s, {Y_{1_i}}^s, {Y_{2_i}}^s\big)
\end{cases}
$$

$$
\begin{cases}
K_2^s = F_1\big(t_i + 0.5h, {X_i}^s + 0.5{K_1}^s h, {Y_{1_i}}^s + 0.5{L_1}^s h, {Y_{2_i}}^s + 0.5{J_1}^s h\big) \\
L_2^s = F_2(t_i + 0.5h, {X_i}^s + 0.5{K_1}^s h, {Y_{1_i}}^s + 0.5{L_1}^s h, {Y_{2_i}}^s + 0.5{J_1}^s h\big) \\
J_2^s = F_3\big(t_i + 0.5h, {X_i}^s + 0.5{K_1}^s h, {Y_{1_i}}^s + 0.5{L_1}^s h, {Y_{2_i}}^s + 0.5{J_1}^s h\big)
\end{cases}
$$

$$
\begin{cases}
K_3^s = F_1\big(t_i + 0.5h, {X_i}^s + 0.5{K_2}^s h, {Y_{1_i}}^s + 0.5{L_2}^s h, {Y_{2_i}}^s + 0.5{J_2}^s h\big) \\
L_3^s = F_2\big(t_i + 0.5h, {X_i}^s + 0.5{K_2}^s h, {Y_{1_i}}^s + 0.5{L_2}^s h, {Y_{2_i}}^s + 0.5{J_2}^s h\big) \\
J_3^s = F_3\big(t_i + 0.5h, {X_i}^s + 0.5{K_2}^s h, {Y_{1_i}}^s + 0.5{L_2}^s h, {Y_{2_i}}^s + 0.5{J_2}^s h\big)
\end{cases}
$$

$$
\begin{cases}
K_4^s = F_1\big(t_i + h, {X_i}^s + {K_3}^s h, {Y_{1_i}}^s + {L_3}^s h, {Y_{2_i}}^s + {J_3}^s h\big) \\
L_4^s = F_2\big(t_i + h, {X_i}^s + {K_3}^s h, {Y_{1_i}}^s + {L_3}^s h, {Y_{2_i}}^s + {J_3}^s h\big) \\
J_4^s = F_3\big(t_i + h, {X_i}^s + {K_3}^s h, {Y_{1_i}}^s + {L_3}^s h, {Y_{2_i}}^s + {J_3}^s h\big)
\end{cases}
$$

The presented equations outline the Modified Runge-Kutta method, a numerical technique for solving a system of differential equations. This iterative method approximates the random numerical solution by updating values based on previous iterations and their derivatives and each value s. It is a vital tool for understanding and simulating random dynamic behaviour in a system described by differential equations 3.

3. Results

In this section, we present the obtained numerical results, providing key insights into the system's dynamic behaviour and highlighting the impact of uncertainty on the predator-prey interactions.

4.1. The statistical properties of random solutions of two predators one prey model

In statistics, a continuous random variable can be described by its probability density function (PDF). The PDF represents the relative likelihood of the variable taking on specific values within a given interval. By examining the PDF, we can determine various statistical properties associated with the random variable, including its mean and variance. The mean represents the expected value or the average outcome, while the variance quantifies the spread or dispersion of the random variable's values around the mean. Both the mean and variance are essential features that provide valuable insights into the characteristics and behaviour of the continuous random variable. [1].

In the context of the ratio-dependent two predators-one prey model, the initial population sizes are assumed to follow a Beta distribution. The model exhibits crisp behaviour for all solutions with initial states within the sets $\{X_0\}$, $\{Y_{0_1}\}$, $\{Y_{0_2}\}$. However, to account for uncertainty, a random solution is ultimately constructed. The Probability Density Function (PDF) of the predators and prey describes their distribution and probabilistic behaviour over time. The PDF of prey and predators is numerically calculated, initially symmetric around their means, and exhibits various shapes as time progresses. Solving the random differential equation involves calculating essential statistical functions associated with the random solution, including the Cumulative Density Function (CDF), mean, and variance functions. Determining the probability density function of the random solution enables us to understand several statistical properties. The behavior of the PDF and CDF of the initial state of the prey class X_0, significantly influences the behavior of the PDF and CDF of the random solution at different time points. This analysis provides valuable insights into the dynamics and uncertainty of the predator-prey system.

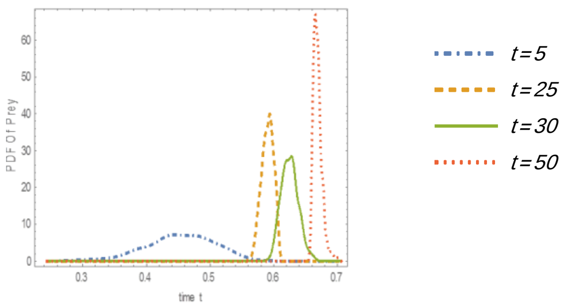

Fig. 2: The behavior of PDF of prey with different values of $\boldsymbol{t}$.

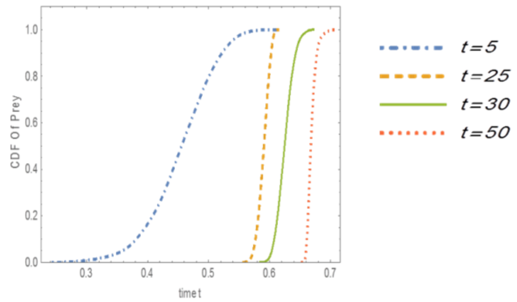

Fig. 3: The behavior of CDF of prey with different values of $\boldsymbol{t}$.

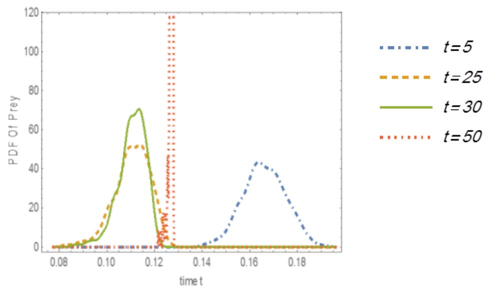

Fig. 4: The movement of the probability distribution functions of prey over time.

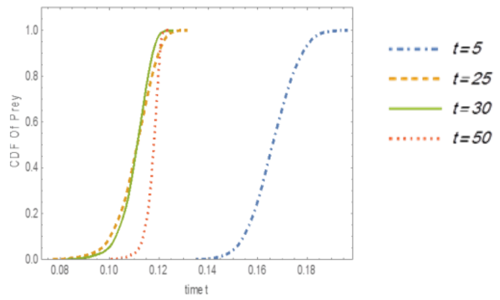

Fig. 5: The behavior of CDF of prey with different values of $\boldsymbol{t}$.

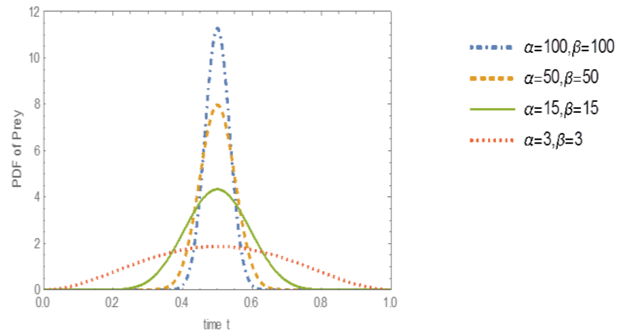

Fig. 6: The CDF of prey with changing the parameters $\boldsymbol{\alpha}, \boldsymbol{\beta}$ at $\boldsymbol{t} = \mathbf{0}$.

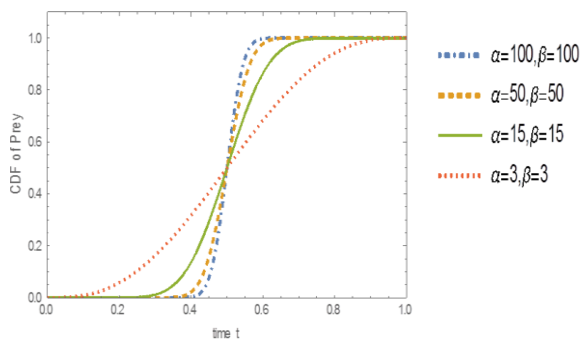

Fig. 7: The CDF of prey with changing the parameters $\boldsymbol{\alpha}, \boldsymbol{\beta}$ at $\boldsymbol{t} = \mathbf{0}$.

4.2. The Mean, Variance and Range of Random Solution

The mean is a fundamental characteristic of the random solution, as it defines the central location of the distribution for both the prey and the predators. Selecting different means, which represent the mean of the Beta distribution, significantly impacts the initial states of the populations and consequently influences the dynamics and behaviour of these populations over time. By altering the mean, we can observe shifts in the distribution of the initial population sizes, leading to varying outcomes in the predator-prey interactions and shedding light on the sensitivity of the ecological system to changes in this essential parameter.

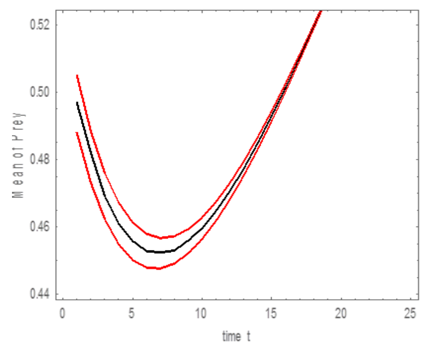

Fig. 8: The behavior of Mean of prey over time.

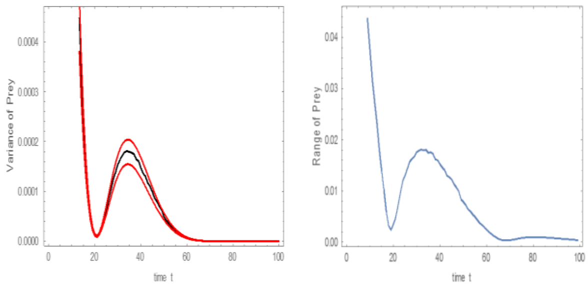

Fig. 9: The behavior of variance, range of prey over time.

Fig. 9: shows a 95% confidence interval for the preys variances, with a different sample size $m = 400$ where the quantities in the simulation are set $X_0 \sim Beta(15,15)$ and. The bold lines in these figures represent the 95% confidence interval limits.

4.1. The Skewness and Kurtosis of prey

The skewness and kurtosis are additional vital properties of the random solution, complementing the understanding of the prey's distribution. Skewness measures the asymmetry of the probability distribution, indicating whether the data is skewed to the left or right of the mean. When the skewness is positive, the distribution is skewed to the right, whereas negative skewness implies a left-skewed distribution. The kurtosis, on the other hand, characterizes the shape of the distribution and quantifies the tails' thickness relative to a normal distribution. Positive kurtosis indicates heavy tails, while negative kurtosis reflects lighter tails compared to the normal distribution. By examining the skewness and kurtosis of the prey's distribution, we gain deeper insights into the underlying probabilistic behaviour and asymmetry of the population dynamics. Moreover, studying how these properties change over time provides valuable information on the evolution and stability of the ecological system under the influence of uncertain initial conditions.

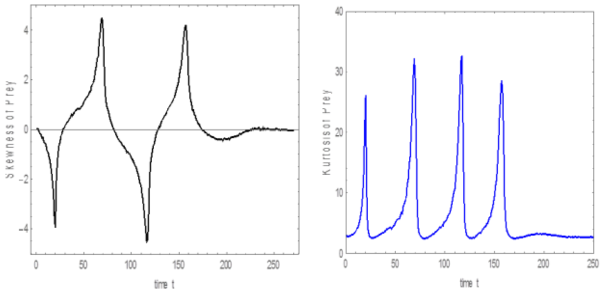

Fig. 10: The behavior of Skewness and Kurtosis of prey over time.

4.2. The Mean, Variance and Range of First Predator

Assuming $Y_1(t) = 0.2$, as a fixed value figures **11-13** will illustrate the dynamic behaviour of various statistical properties within a 95% confidence interval for the 1st predator over time. Specifically, the figures will depict the mean, variance, range, skewness, and kurtosis of the predator's population within this confidence interval as time progresses. By presenting these statistical measures over time, the analysis provides valuable insights into the evolving trends and fluctuations in the 1st predator's population dynamics, considering the inherent uncertainty associated with the 95% confidence interval. This extended investigation sheds light on the sensitivity and variability of the predator's population, further enhancing our understanding of the ecological dynamics in the given model.

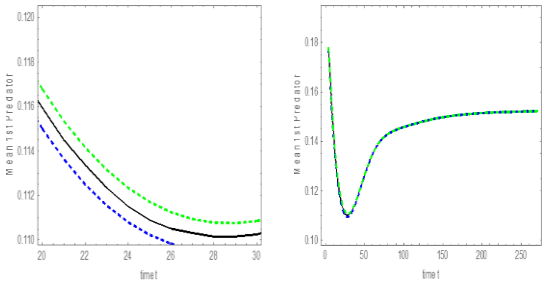

Fig. 11: Changing the Mean of the 1st predator over time.

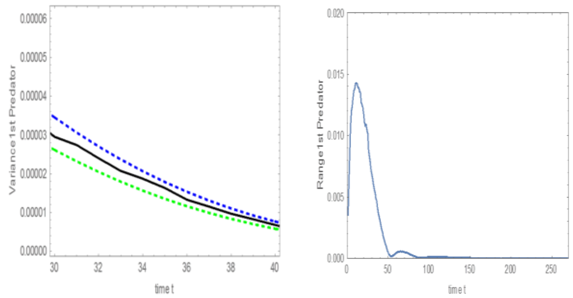

Fig. 12: Changing the variance, range of 1st predator over time.

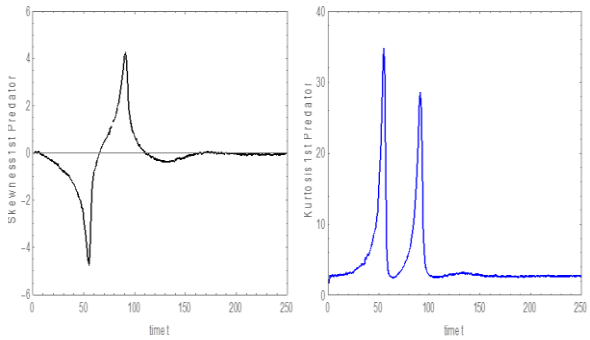

Fig. 13: The behavior of Skewness and Kurtosis over time.

4.1. The Mean, Variance and Range of Second predator

The figures **14-16**, which delve into the dynamic behavior of the 2nd predator within a 95% confidence interval over time. These figures portray key statistical properties, including the mean, variance, range, skewness, and kurtosis, all of which offer valuable insights into the fluctuations and trends characterizing the 2nd predator's population dynamics. By exploring this uncertainty within the confidence interval, we gain a deeper understanding of how the 2nd predator's population responds and adapts over time, unraveling the intricate complexities of the ecological interactions at play. This comprehensive analysis not only enhances our knowledge of the model's behavior but also sheds light on the broader implications of uncertainty in ecological systems.

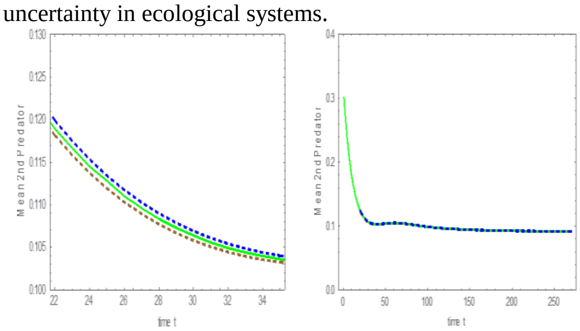

Fig. 14: Changing the Mean of the 2nd predator over time.

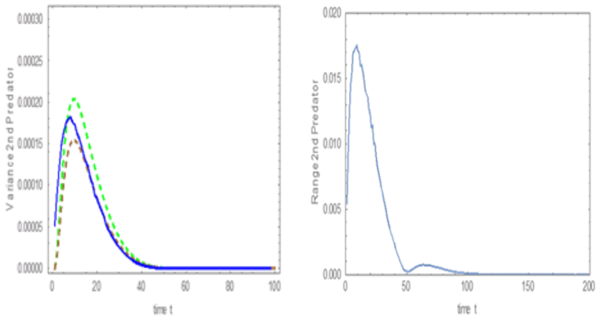

Fig. 15: Changing the variances, range of the 2nd predator over time.

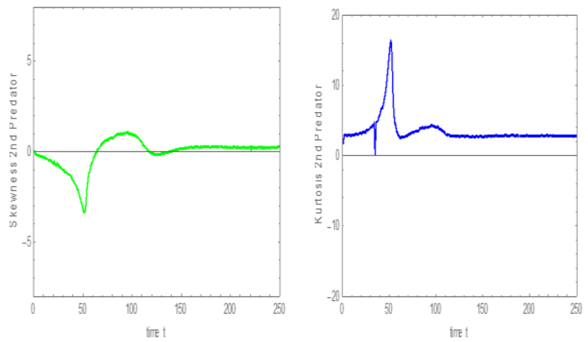

Fig. 16: The behavior of Skewness and Kurtosis over time.

4. Discussion

In the ratio-dependent two predators-one prey model **4**, the mean of the random solution for predators and prey generally exhibits a crisp behavior. Additionally, the 95% confidence intervals are calculated for both predator and prey means, providing a range within which the true mean μ is expected to lie with a 95% confidence level. Conversely, in a 5% confidence interval, this certainty does not hold.

Furthermore, in the same model **4** that is dependent on ratios, the initial variance for both predators and prey aligns with the variance of the chosen Beta distribution, which serves as the initial state for prey populations. This variance fluctuates over time, indicating oscillations in the width of the probability density function describing the random solution. Despite these oscillations, the variance remains finite over time.

The results illustrate diverse shapes taken by the density probability functions of the three classes over time. The variability in α and β significantly influences the speed and direction of the density functions' shift, either towards left or right skewness.

Notably, the behavior of the probability density functions and the moments of density functions are contingent upon the values of α and β. Additionally, the cumulative distribution functions are numerically obtained for both predators and prey. These curves help infer whether the distribution has a low or high degree of kurtosis, providing insights into the spread of prey and their predators.

Conclusion

This research has centred on investigating the role of uncertainty in a two predators-one prey model with a ratio-dependent functional response. Specifically, the uncertainty have been explored surrounding the initial value of the prey population. Our primary objective has been to address the challenges posed by dealing with and quantifying uncertainty in ecological systems. To tackle this issue effectively, the beta distribution was employed as a powerful tool. By treating it as a random initial state for the prey population, it could account for the uncertainty that affects the final size of the model. Through the analysis, we examined how the parameters of the beta distribution influence the model's behaviour. The results indicate that random behaviour represents a more realistic and generalized approach compared to the classical crisp behaviour, enriching the description of the ecological phenomenon. This study contributes valuable insights into the complex dynamics of predator-prey interactions under uncertain conditions, paving the way for a deeper understanding of ecological systems' behaviours.

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