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The Variable Mass- Spring System and the Bessel's Function

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ABSTRACT

Damped harmonic oscillator and Bessel's function both exhibit similar behaviours. In order to justify this similarity between the two functions, the mass-spring system with variable mass is used. It is shown how the damped harmonic oscillator can be used as an approximated function for the Bessel function. The adiabatic invariance is also employed to make the mathematical and physical comparison between the two functions. It is also shown how to use the zeros of the Bessel's function to obtain the same approximation by solving the boundary-value problems. The zeros of the Bessel's function are used to calculate the time period of the variable mass-spring system.

نظام الكتلة المتغيرة-النابض و دالة بيسل

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قسم الفيزياء، كلية العلوم ، جامعة عمر المختار ، البيضاء ، ليبيا

الكلمات المفتاحية:
نظام كتلة-نابض
دالة بيسل
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كتلة متغيرة
ثبات أدياباتي.

الملخص

المتذبذب التوافقي المخمد و دالة بيسل كلاهما يظهر سلوك متشابه. ولكي نبرر هذا التشابه بين الدالتين تم استخدام نظام الكتلة-النابض بكتلة متغيرة. تم بيان كيف أن المتذبذب التوافقي المخمد يمكن استخدامه كدالة تقريبية لدالة بيسل. الثبات الادياباتي ايضا تم استخدامه لأجل المقارنة الرياضية والفيزيائية بين الدالتين. تم ايضا بيان كيفية استخدام أصفار دالة بيسل للحصول على نفس التقريب بحل مسائل القيمة الحدية. استخدمت أصفار دالة بيسل لحساب زمن الفترة لنظام الكتلة المتغيرة-النابض.

Introduction

Due to the importance of oscillatory motion in a wide range of application in real life, such engineering applications, necessitates studying the mass-spring system with variable mass and probing its dynamics through computational calculations of the physical properties of this system which is dependent essentially on harmonic motion. The variable mass-spring system belongs to what is so called the variable mass system, one example of the integration of Newton's equation of motion. It was assumed that the particle mass to remain constant. This assumption was according to the fact that in most problems in classical mechanics, velocities are but a small fraction of the speed of light. Hence, the relativistic effects may be ignored. However, a particle may change its mass with time, such as a raindrop losing mass by evaporation or gaining mass by condensation, or a rocket in the short time interval during which its fuel is being consumed [1-12].

Bessel's functions are solutions for what is known as Bessel's differential equation. This kind of functions was used by Bessel, a German mathematician and astronomer, in dealing with problems in dynamical astronomy. The importance of this equation and its solutions (Bessel functions) comes from the fact that they appear frequently in the boundary-value problems of mathematical physics and engineering. The Bessel equations are also concerned with problems of cylindrical symmetry and many others. Hence, Bessel functions are sometimes called cylindrical functions [13]. The variable mass-spring system behaves like a damped harmonic oscillator due to the variation of the system's mass. The decaying in amplitudes of the oscillator due to the presence of some dissipative forces is called damping. These oscillations with decreasing amplitude are called damped oscillations. However, in the case of variable mass-spring there are no such frictional forces to resist the motion of the system, but the behaviour is similar to that of the damped harmonic oscillator [14]. Bessel functions arise as solutions for the differential equations known as Bessel's differential equations. It is not easy to deal with problems that consist of Bessel functions due to their long expressions and complexity. Fortunately, the damped harmonic oscillator behaves as the Bessel functions do. There are very noticeable similarities between the curves of the damped harmonic functions and Bessel functions. Hence, any physical problem that involves Bessel functions can be treated by approximating it through the damped harmonic functions. The aim of this paper is to verify the possibility of approaching Bessel functions by the damped harmonic by showing the basic differences or intersections between them and discussing some of the dynamical properties of the studied system like position, energy and power. For

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the systems where Bessel functions appear, the equations of motion behave as a damped system with decaying amplitude but with a variable period of oscillation. A parametrically excited system, like the variable mass-spring system, is used as a physical model to investigate where the two set of functions agree. Then, the results of the exact solution that contain Bessel functions are studied and approximated with the solution of the damped harmonic oscillator [15]. The zeros of Bessel functions play an important role in computational mathematics, mathematical physics, and other areas of natural sciences [16]. Through the zeros of the Bessel's function, one can find the values of the physical parameters that lead to the approach between the Bessel's function and the damped harmonic function.

The Damped Variable Mass–Spring System and Bessel's Function

Consider a one dimensional physical oscillator that consists a massless spring attached to a rigid wall while the other end is attached to an object of mass (m) linearly changing with respect to time according to the following relation $m(t) = m_0 + \gamma t$ where m_0 is the initial mass and ($\gamma > 0$) is rate at which the mass increased. To derive the equation of motion that governs the dynamical response of the mass–spring system as its mass varies with time [15], Newton's second law will be used as follows [12]:

$$\frac{d}{dt}(mv) = \sum F \quad (1)$$

where the force acting on the system is the restoring force $-kx$ that pushes the system toward the equilibrium position, hence equation (1) becomes:

$$\frac{dm}{dt}v + m\frac{dv}{dt} = -kx$$

Using the notation of $v = \frac{dx}{dt}$, then the above equation becomes:

$$m\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + kx = 0 \quad (2a)$$

Or in another form as:

$$\frac{d^2x}{dt^2} + 2\beta(t)\frac{dx}{dt} + \omega_0(t)x = 0 \quad (2b)$$

Where $\beta(t) = \frac{\gamma}{2m}$and $\omega_0(t) = \frac{k}{m}$ .This kind of differential equation is called parametric differential equation which arises in physics in what so call a parametric excitation where one of the system's parameter is changing with time due to external exciter. A physical system undergoes a parametric forcing if one of its parameters is modulated periodically with time. A common familiar example of parametric excitation of oscillations is given by the playground swing on which most people have played in childhood [13].

Since the change in mass will effects on the motion of the system, hence one has to change the dependency of equation (2a) from time (t) into mass (m) i.e. by using chain rule:

$$\frac{dx}{dt} = \frac{dm}{dt}\frac{dx}{dm} = \gamma\frac{dx}{dm}$$

and

$$\frac{d^2x}{dt^2} = \gamma^2\frac{d^2x}{dm^2}$$

Therefore, putting these expressions into equation (2a), one obtains:

$$m^2\frac{d^2x}{dm^2} + m\frac{dx}{dm} + \frac{k}{\gamma^2}mx = 0 \quad (3)$$

The obtained differential equation is much more similar to the following differential equation:

$$x^2\frac{d^2y}{dx^2} + (2k+1)x\frac{dy}{dx} + (\alpha^2x^{2r} + \beta^2)y = 0 \quad (4)$$

This is known as the transformable Bessel's differential equation. Then by making a comparison with the differential equation (3), one gets:

$$2k+1 = 1 \rightarrow k = 0\,, \alpha^2 = \frac{k}{\gamma^2}\,, r = \frac{1}{2} \text{ and } \beta^2 = 0$$

The general solution for the angular position function of the obtained differential is:

Hence, the angular position function is:

$$x(t) = \frac{\pi\sqrt{km_0}}{\gamma}x_0[RY_0(\frac{2\sqrt{k}}{\gamma}(m_0+\gamma t)^{\frac{1}{2}}) - QJ_0(\frac{2\sqrt{k}}{\gamma}(m_0+\gamma t)^{\frac{1}{2}})] \quad (5)$$

Or

$$x(t) = D\alpha(t)$$

where $\alpha(t) = RY_0\left(\frac{2\sqrt{k}}{\gamma}(m_0+\gamma t)^{\frac{1}{2}}\right) - QJ_0\left(\frac{2\sqrt{k}}{\gamma}(m_0+\gamma t)^{\frac{1}{2}}\right)$,

R = $J_1(\frac{2\sqrt{km_0}}{\gamma})$, Q = $Y_1\left(\frac{2\sqrt{km_0}}{\gamma}\right)$ and $D = \frac{\pi\sqrt{km_0}}{\gamma}x_0$ are constants [7].

Returning to the differential equation (2b), if the case of the adiabatic change (slow change) is considered, then the length $m(t) \rightarrow m_0$ and therefore both of $\beta(t)$and$\omega_0(t)$ become constants hence, the differential equation (2b) seems like an equation of damped harmonic oscillator and so the solution will be for the case of under damped oscillation where ($\beta^2 < {\omega_0}^2$) as $x(t) = Ae^{-\beta(t)t}\cos(\omega(t)t - \delta)$ where $\omega^2 = {\omega_0}^2 - \beta^2$ is the angular frequency of oscillation. At small time period ($\Delta t = t_{n+1} - t_n$), the solution at $t = t_{n+1}$ will be related to that one at $t = t_n$, hence:

$$x(t_{n+1}) = Ae^{-\beta(t_n)\Delta t}\cos(\omega(\Delta t) - \delta)$$

This expression shows how the Bessel function can be approximated by damped harmonic oscillator function [16]. The equation of motion for the variable length pendulum can also be written with respect to $y = \frac{2\sqrt{k}}{\gamma}(m_0+\gamma t)^{\frac{1}{2}}$ such that

$$\frac{d^2x}{dy^2} + 2\beta(y)\frac{dx}{dy} + x = 0 \quad (6)$$

where $\beta(y) = \frac{1}{2y}$is the damping factor. Under the condition of adiabatic change the solution for the above differential equation will be approximated at y_m by:

$$x_m \approx A_m e^{-\beta_m y}\cos(\omega_m y - \vartheta_m) \quad (7)$$

where$\beta_m = \frac{1}{2y_m}$ and $\omega_m = \sqrt{1 - {\beta_m}^2}$. Consider that the problem of the variable length problem will be solved precisely by using some initial conditions. The solution is obtained by Bessel functions at some given value for y_m such that the perfect conditions of the motion, (x and$\frac{dx}{dt}$) can be found and inserted into equation (7) to obtain (A_m) and (ϑ_m) [4. Therefore, one gets:

$$x_m \approx A_m e^{-\beta_m y_m}\cos(\omega_m y_m - \vartheta_m) \quad (8)$$

And the velocity $\dot{x}_m$ at $y = y_m$ is:

$$\dot{x}_m = -A_m e^{-\beta_m y_m}(\beta_m\cos(\omega_m y_m - \vartheta_m) + \omega_m\sin(\omega_m y_m - \vartheta_m)) \quad (9)$$

From equation (8), one has:

$$A_m e^{-\beta_m y_m} = \frac{x_m}{\cos(\omega_m x_m - \vartheta_m)}$$

Then equation (9) becomes:

$$\dot{x}_m = -x_m(\beta_m + \omega_m\, tan(\omega_m y_m - \vartheta_m))$$

Hence, solving for (ϑ_m), one gets:

$$\vartheta_m = \omega_m y_m - tan^{-1}\left(-\left(\frac{\dot{x}_m + \beta_m x_m}{\omega_m x_m}\right)\right)$$

For (A_m), one obtains:

$$A_m = \frac{\sqrt{(\dot{x}_m + \beta_m x_m)^2 + (\omega_m x_m)^2}}{\omega_m}e^{\beta_m y_m}$$

Solving equation (9) and the related equations with-it constitute a model when determined numerically allows for the comparison with Bessel functions.

Results and Discussion

For the problem of the variable mass-spring system, x and m are the two independent variables. The variation occurs in the linear momentum of the system is due to variable mass since the linear momentum depends directly on the mass of the system i.e. $P = mv$. The change of the momentum leads to the appearance of the velocity term in the equation of motion i.e., dx/dt which make the problem similar to the damped case. In mathematics, the first derivative in the second order differential equation works on suppressing the system and the term multiple by that first derivative is called the damping term which leads to the decaying of the system's amplitudes. In fact, and in the absence of any kind of dissipated forces, like frictional forces, the variable mass-spring system behaves as a non-conservative system where its energy is time-varying.

Figure 1 shows the curve of both the suggested model that consists of Bessel function and the approximated damped harmonic function for the linear displacement. While Figure 2 presents the exact solution involving Bessel functions and the approximated damped harmonic oscillator compared with that was published previously by Masoud [15]. They both reveal a very good agreement between the exact solution that involves Bessel function (suggested model) and the approximated damped cosinosoidal function (damped oscillator) for a small-time interval. The two solutions agree very well at some physical values of the mass-spring system i.e., at the spring constant (k) and the rate of mass increasing (γ). The two solutions were evaluated at some specific point $y_m = 100$, and exactly at this point, the two curves of the suggested and damped functions are completely close to each by the constraints at y_m. In other words, at ($x_m, \dot{x}_m$) which are evaluated from the exact solution and then inserted into the damped harmonic function, this substitution shows a matching between the two functions.

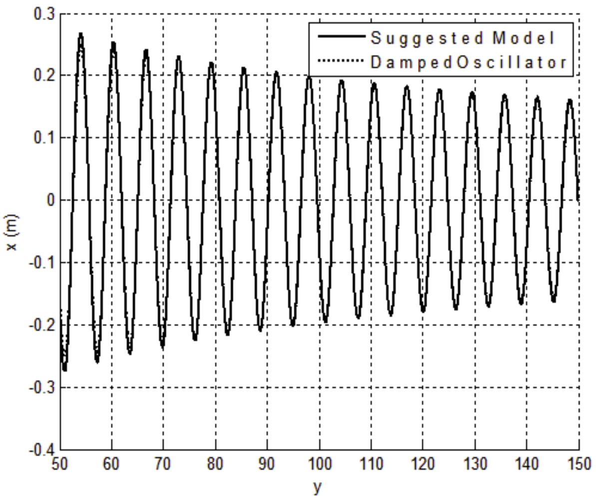

Figure 1: The exact solution involves Bessel functions (solid line) and the approximated damped harmonic oscillator (dotted line) at the following values for the variable mass-spring system and damped oscillator: m_0 = 1 kg, x_0 = 1 m, k = 100 N/m, γ = 2.25 g/s, y_m = 100, x_m = -0.0496 m, $\dot{x}_m$ = - 0.1915 m/s for the linear displacement.

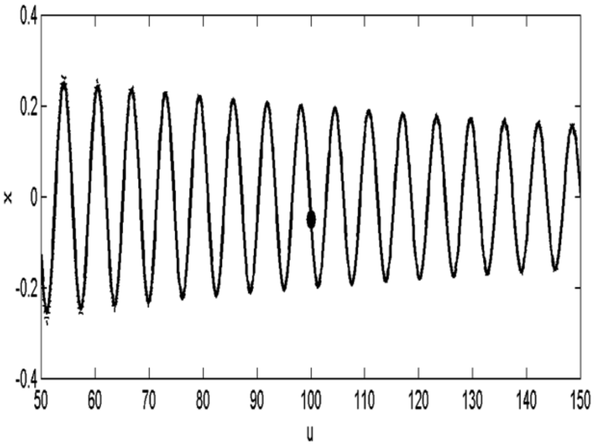

Figure 2: The exact solution involving Bessel functions (solid line) and the approximated damped harmonic oscillator (dotted) at the following values for the variable mass-spring system and damped oscillator: m_0 = 1 kg, x_0 = 1 m, k = 100 N/m, γ = 2.25 g/s, y_m = 100, x_m = - 0.0496 m, $\dot{x}_m$ = - 0.1915 m/s for the linear velocity. The solution corresponding to u_n= 100 is shown by a dot on the graph [15].

The approximation that occurred between the Bessel function and the damped harmonic function was done according to the physics of the variable mass-spring system. In comparison with the work introduced by Masoud [15], where the approximation was achieved through some mathematical assumptions by suggesting that the spring constant (k) and the rate of mass increase (γ) are related to each other such that the parameter $y_0 = \frac{2\sqrt{km_0}}{\gamma}$ represents the first root for the Bessel function J_0. Through the y-parameter and by giving some values for the hook's constant (k), one can find the values of the rate of mass increase that leads to the agreement between the two functions. If one chose k to be a complete square then:

$$k = n^2$$

Hence, and for m_0 = 1 kg the values of γ are given by:

$$\gamma_n = \frac{2n}{y_0}$$

Where n = 1, 2, 3………..

Figure 3 shows the variation of the linear displacement of the variable mass-spring system as a function of time at the following data: m_0 = 1 kg, k = 15 N/m, x_0 = 1 m, γ = 1 m/s. As it was shown in Figure 1, the damped harmonic function can be used as an approximated function for the problems that involves Bessel function, so the analysis and description for the behaviour of the variable mass-spring system will be as for the damped harmonic. From this figure, it can be seen that as time passes the amplitude of oscillation decreases as the rate of mass increases. It can be also observed that the magnitudes of the amplitudes decreased slowly as time passes, which means that the variable mass-spring system takes a very long time to get to rest.

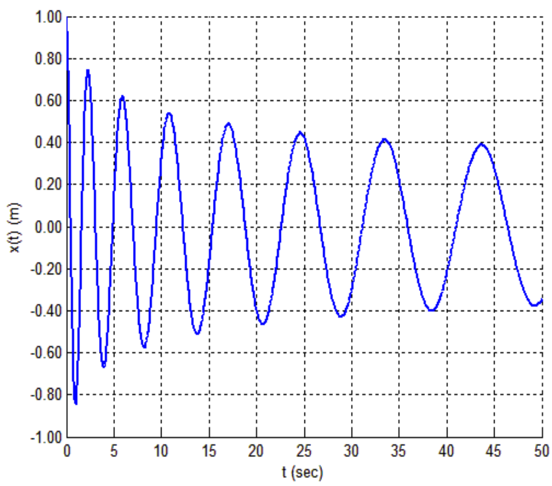

Figure 3: The linear displacement x(t) of the variable mass-spring

system at the following data: m_0 = 1 kg, k = 15 N/m, x_0 = 1 m, γ = 1 g/s.

Figure 4 shows the change of the angular frequency of the variable mass-spring system as a function of time. One can see that the angular frequency, at which the system oscillates, decreases as time runs. Due to the dependency of the time period of oscillation on the angular frequency of the motion, it can be deduced that the decrease in the angular frequency leads to the increase in the time period of motion, therefore as the rate of mass increases; the time taken by the oscillator to perform one complete cycle also increases. Hence, the system will take a very long time to come to the equilibrium position.

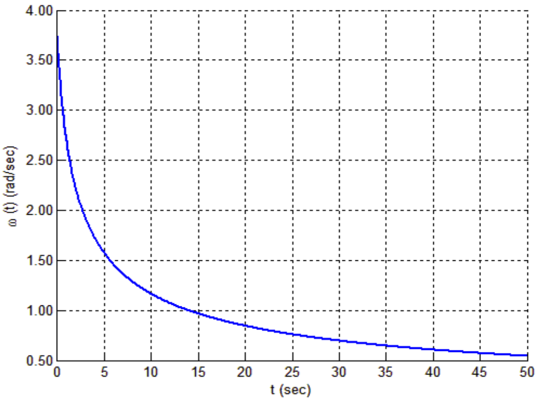

Figure 4: The angular frequency ω(t) of the variable mass-spring system at the following data: m_0 = 1 kg, k = 15 N/m, x_0 = 1 m, γ = 1 g/s.

Figure 5 shows the change of the time period as a function of the stiffness constant at fixed rate of mass increase of γ = 0.2 g/s. According to this figure, one can notice that as the stiffness constant increases, the time period of the system decreases, i.e. the time period changes inversely with respect to the stiffness constant. The decrease in the values of the system's time period means that the action of increasing the stiffness constant is to run the system as fast as possible to finish its oscillation at the shortest possible period of time because increasing the stiffness constant leads to increasing the elastic potential energy of the system which is converted into a kinetic energy and increases it.

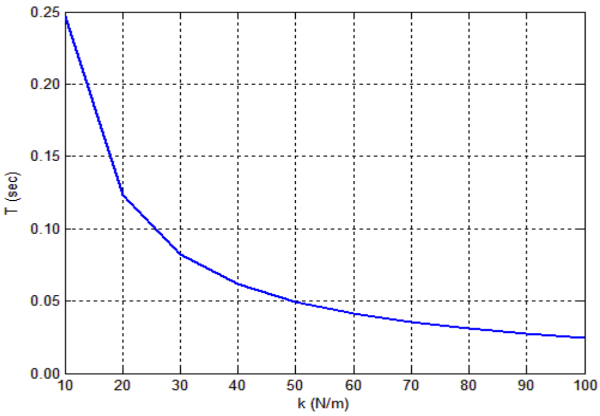

Figure 5: The variation of the time period T as a function of the stiffness constant k at γ = 0.2 g/s and at the first cycle n = 1.

5. Conclusion

The curves of Bessel and damped harmonic functions are somehow identical and agree with each other. To prove the agreement between the Bessel and damped harmonic functions, the variable mass-spring system is presented as a model. This system includes in its equation of motion the Bessel functions that have to be approximated by the damped harmonic oscillator function. The two functions (Bessel and damped harmonic) were compared to each other at some specific point y_m = 100. The obtained results showed that the approximation between the two solutions at this particular point with that of the damped harmonic oscillator to be a very good approximate function for the Bessel function for large values of y i.e. $y \geq y_m$. It was also found that, even without restricting the problem for the purpose of simplicity by inserting some conditions in order to reduce the calculations, the unrestricted model was also in a good agreement with the damped harmonic function. It was deduced that the agreement between the two functions also occurred at some particular values of both (k) and (γ). It was also concluded the values of the rate of the mass increase provided the matching between the two functions at some fixed value for the spring constant (k) through using the Bessel's zero.

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