The Quadratic Trigonometric Bezier like Curve With two Shape Parameters

*Mabrokah Aboulqsim Mohammed kilani, Almbrok H. A. Omar, Iman I. A. Ahmed, Fatima Alzahraa Mohammed Abuzaiyan

Mathematics Department, Faculty of Sciences - Sebha University, Libya

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**A B S T R A C T**

The quadratic trigonometric Bezier model serves as a potent and valuable tool in computer-aided geometric design, boasting exceptional geometric properties. In this study, Novel functions are introduced by leveraging a trigonometric Bezier-like curve with adjustable shape parameters. Each curve segment is formed by three consecutive control points, providing precise control over shape by adjusting these parameters while keeping the control polygon unchanged. These curves exhibit superior proximity to the control polygon compared to quadratic Bezier curves across all shape parameter values, gradually approaching the control polygon as the shape parameter increases. Moreover, for practical application, the quadratic trigonometric Bezier-like curve was utilized to artfully design the Arabic word “الله” (Allah) and define flower contours, employing MATHEMATICA software. The results underscore the model’s effectiveness in accurately representing intricate shapes, offering a versatile and efficient solution for diverse design challenges. Furthermore, the fusion of the quadratic trigonometric Bezier-like curve with shape parameters significantly augments design flexibility and adaptability, making it a valuable asset in computer graphics, geometric modeling, and artistic design across multiple domains.

**المستقبل**

يعتبر النموذج التربيعي المثلثي لبيزيير أداة قوية وقيمة في تصميم الهندسة المعززة بالحاسوب، حيث يتميز بخصائص هندسية ممتازة. في هذه الدراسة، نقدم وظائف جديدة باستخدام منحنى يشبه بيزيير المثلثي مع معلمات الشكل قابلة للتعديل في مجال محدد. بحيث يتيح لكل منحنى من ثلاث نقاط تحكم مماثلة، مما يتيح لنا التحكم بشكل دقيق في شكل المنحنى عن طريق ضبط معلمات الشكل وحسب حاصل على مضلع التحكم نابئًا. تطور هذه المنحنى تقاربًا متقنًا لمضلع التحكم مقارنة بمنحنيات بيزيير التربيعية عبر جميع معلمات الشكل، مع التبادل اليدوي للمنحنى نحو مضلع التحكم في مجال المحدد. وعلاوة على ذلك، كتطبيق عالي، استخدمنا منحنى بيزيير المثلثي الشكل لتصميم كلمة “الله” بشكل ماهر وتحديد الشكل الخارجي للزهرة، باستخدام برنامج فعالية هذا النموذج في تمثيل الأشكال المعقدة بدقة، مما يوفر حلاً متعدد الاستخدامات وفعالًا لتحديات التصميم المتنوعة. علاوة على ذلك، فإن مسح منحنى بيزيير المثلثي الشكل مع معلمات الشكل يعزز بشكل كبير مرونة وقابلية التكيف في عملية التصميم، مما يجعله أداة قيمة في مجالات متعددة، بما في ذلك الرسومات الحاسوبية والنموذج الهندسية والتصميم الفني.

1. **Introduction**

The paper In curve design, users prefer easily understandable and manipulable curves. The influence of artistic design on the development of free-form curves is notable. A well-known type is the curved Bezier, initially intended for automotive body design and now a captivating area of applied mathematics. Bezier curves, rooted in Bernstein’s borders, find wide-ranging applications in computer...
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The main results

1. Preliminary concepts

In this section, the fundamental concepts essential for understanding the topic will be introduced. These concepts serve as a foundation for the ensuing discussion, elucidating key terms and theories.

2.1. Bernstein basis

The general formula of the (th Bernstein polynomials is defined as

\[ B_{i,n}(t) = \binom{n}{i} t^i (1-t)^{n-i}, \quad t \in [0,1]. \]

For \( i = 1, 2, \ldots, n \), where \( n \) is the degree of the curve [5].

2.2. Control Points

The precise positioning of control points significantly influences the curve's shape. While the start and end points can be determined, the middle control point remains more flexible, as different middle points can produce the same curve with identical start and end points. Hence, there is no local control point in this regard. To achieve desired variations in the curve, the shape parameters need to be altered, the selection of corner points and the determination of segment numbers further impact the curve's appearance. Moreover, the curve can be easily modified by adding or repositioning the control points. These factors collectively contribute to the versatility and adaptability of the curve, empowering designers with creative control over the final outcome [6].

A control polygon serves as a vital tool in shaping objects, consisting of a sequence of control points arranged in space. When these control points are connected by straight lines, the resulting polygon is referred to as the control polygon. This geometric construct holds significant importance in computer graphics, design, and various modeling applications. By manipulating the positions of the control points, designers can easily alter the shape of the object or curve, allowing for creative and intuitive control over its final form. The control polygon provides a visual representation of how the control points influence the object's appearance, making it easier to understand and adjust the design. It serves as a fundamental element in many geometric modeling techniques, facilitating the creation of smooth curves, surfaces, and shapes in diverse fields such as computer-aided design (CAD), computer graphics, and animation [7].

2.3. Bezier Curve

The Bezier curve is a fundamental concept in the field of computer graphics and geometric modeling. It was introduced by Pierre Bezier in the 1960s and has since become a widely used technique for designing smooth curves and surfaces. Bezier curves are defined by a set of control points that dictate the curve's shape and behavior. With their intuitive control and versatility, Bezier curves find applications in various domains, including computer-aided design, animation, and font creation. In this introduction, the key properties and applications of Bezier curves are explored, shedding light on their significance in modern design and visualization. The Bezier curve defined by control points \( P_i \in \mathbb{R}^n \), is the \( n \)-th degree polynomial given by

\[ r(t) = \sum_{i=0}^{n} B_{i,n}(t) P_i \quad \text{for} \ t \in [0,1]. \]

The multipliers at the points in the above equation are called Bernstein basis and are defined as

\[ B_{i,n}(t) = \binom{n}{i} t^i (1-t)^{n-i}. \]

Figure 1: Bezier curve of degree 2.

2.4. The shape parameters

Let three \( P_0, P_1, P_2 \) control points, a family of generalized quadratic trigonometric Bezier-like curves can be generated by the shape parameters \( \alpha, \beta \) take on different values in their respective value ranges. Due to the fact that each generalized quadratic trigonometric Bezier-like curve provides three shape parameters, the shape of a quadratic trigonometric Bezier-like curve can be adjusted flexibly by modifying its shape parameters [7].

2. The main results

In our endeavour to enhance the versatility of Bezier curves for practical applications, the aim is to derive new functions that
combine quadratic trigonometric and linear polynomial basis functions. These innovative functions will play a pivotal role in our various applications. To achieve this, shape parameters, denoted as $\alpha$ and $\beta$, are introduced, enabling us to gauge the influence of the control points on the curve's shape. By leveraging these shape parameters, the curves can be precisely controlled and tailored to meet specific design requirements, providing us with a powerful and flexible tool for curve manipulation and customization.

**Definition 1**
For two arbitrarily selected real values of $\alpha$ and $\beta$, where $\beta \in [0, 1]$, the following four functions of $t$ ($t \in [0, 1]$) are defined as quadratic trigonometric Bezier basis functions with two shape parameters $\alpha$ and $\beta$:

$$r(t) = \sum_{i=0}^{2} p_i B_{i,2}(t), \quad t \in [0, 1], \quad \alpha, \beta \in [0, 1] \quad (1)$$

where

$$B_{0,2}(t) = (1 - at) \cos \left(\frac{\pi t}{2}\right)^2,$$
$$B_{1,2}(t) = \frac{1}{2} [\beta + (\alpha - \beta)t + (\beta - (\alpha + \beta)t) \sin(\pi t)]$$
$$B_{2,2}(t) = (1 + \beta(t - 1)) \sin \left(\frac{\pi t}{2}\right)^2$$

**Figure 2:** Combination quadratic trigonometric and linear polynomial basis functions with $\alpha = 0, \beta = 0$.

**Figure 3:** Combination quadratic trigonometric and linear polynomial basis functions with $\alpha = 1, \beta = 1$.

**Figure 4:** Combination quadratic trigonometric and linear polynomial basis functions with $\alpha = -1, \beta = -1$.

**Figure 5:** Combination quadratic trigonometric and linear polynomial basis functions with $\alpha = 0, \beta = -1$.

Figures 1, 2, 3 and 6 showcases the combination of quadratic trigonometric and linear polynomial basis functions plotted as a function of $t$, ranging from 0 to 1. A significant characteristic of these polynomials is that, for any value of $t$ within the interval $[0, 1]$, the sum of the functions evaluated at $t$ is always 1. The graph, in conjunction with equation (1), reveals the varying importance of the three control points as $t$ changes. At $t = 0$, only $p_0(t)$ is nonzero, so only the point $p_0$ has to influence the curve segment, $r(t)$. At $t = 1$, the point $p_2$ takes over while the influence of the other control points disappears. In essence, $p_0$ and $p_2$ set the starting and ending points, respectively, of the curve segment. Once $p_0$ is established, the control point $p_1$, along with $p_2$, sets the direction of the curve as it leaves $p_0$. Similarly, the control point $p_1$, along with $p_2$, sets the direction of the curve as it approaches the ending point, $p_2$. Thus, with just three control points, it becomes possible to independently control the starting and ending directions of the curve, providing a powerful and concise means of curve manipulation.

The remarkable flexibility and efficiency of a quadratic trigonometric Bezier-like curve as a design tool stem from its unique ability. This distinct characteristic enables the curve to attain a wide range of shapes and behaviors, making it highly adaptable to various design requirements. These exceptional properties will be demonstrated and validated in the ensuing theorem.

**Theorem 1**

The new basis functions of degree 2, associated with the shape parameters $\alpha, \beta$, have the following properties:

i. Non-Negativity: $B_{i,2}(t) \geq 0, \quad t = 0, 1, 2$

ii. Normalization: $\sum_{i=0}^{2} B_i(t) = 1$

iii. Symmetry: $B_{i,2}(t; \alpha, \beta) = B_{2-i,2}(1 - t; \alpha, \beta), \text{for } i = 0, 1, 2$

Their Bezier-like has the following properties:

$$r(0) = p_0 \quad (3)$$
$$r(1) = p_2 \quad (4)$$
$$r'(0) = \alpha(p_1 - p_0) \quad (5)$$
$$r'(1) = \beta(p_2 - p_3) \quad (6)$$
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Proof

1) For \( t \in [0,1] \) and \( \alpha, \beta \in [0,1] \), then

\[
\left(1 - \alpha t\right) \cos^2 \left(\frac{\pi_t}{2}\right) \geq 0,
\]

because when \( \alpha = 0 \) then

\[
\cos^2 \left(\frac{\pi_t}{2}\right) \geq 0
\]

and when \( \alpha = 1 \) then

\[
\left(1 - t\right) \cos^2 \left(\frac{\pi_t}{2}\right) \geq 0
\]

therefore, in both last functions when \( t = 1 \) they will be 0 and when \( t = 0 \) they will be 1.

2) \( \left(1 + \beta(t - 1)\right) \sin^2 \left(\frac{\pi_t}{2}\right) \geq 0 \), because when \( \beta = 0 \) then

\[
\sin^2 \left(\frac{\pi_t}{2}\right) \geq 0,
\]

and when \( \beta = 1 \)

\[
\left(1 + (t - 1)\right) \sin^2 \left(\frac{\pi_t}{2}\right) \geq 0
\]

then in both last functions when \( t = 1 \) they will be 0 and when \( t = 0 \) they will be 1.

3) \( \frac{1}{2}\beta + (\alpha - \beta)t + (\beta - (\alpha + \beta)t) \sin(\pi t) \)

\[
\left(1 + \beta(t - 1)\right) \sin^2 \left(\frac{\pi_t}{2}\right) \geq 0,
\]

and when \( \beta = 1 \)

\[
\left(1 + (t - 1)\right) \sin^2 \left(\frac{\pi_t}{2}\right) \geq 0
\]

and when \( \beta = 0 \)

\[
\left(1 + \beta(t - 1)\right) \sin^2 \left(\frac{\pi_t}{2}\right) \geq 0,
\]

then in both last functions when \( t = 1 \) they will be 0 and when \( t = 0 \) they will be 1.

4) \( \frac{1}{2}\beta + (\alpha - \beta)t + (\beta - (\alpha + \beta)t) \sin(\pi t) \)

\[
\left(1 + \beta(t - 1)\right) \sin^2 \left(\frac{\pi_t}{2}\right) \geq 0,
\]

because when \( \alpha = 0 \) and \( \beta = 0 \), then

\[
1 - \left(1 - \alpha t\right) \cos^2 \left(\frac{\pi_t}{2}\right) - \left(1 + \beta(t - 1)\right) \sin^2 \left(\frac{\pi_t}{2}\right) \geq 0,
\]

and when \( \alpha = 1 \) and \( \beta = 1 \)

\[
1 - \left(1 - t\right) \cos^2 \left(\frac{\pi_t}{2}\right) - \left(1 + (t - 1)\right) \sin^2 \left(\frac{\pi_t}{2}\right) \geq 0,
\]

then in both last functions, when \( t = 1 \) they will be 0 and when \( t = 0 \) they will be 0.

5) \( \sum_{i=0}^{1} B(t) = 1 \).

\[
\left(1 - \alpha t\right) \cos^2 \left(\frac{\pi_t}{2}\right) + \frac{1}{2}\beta + (\alpha - \beta)t + (\beta - (\alpha + \beta)t) \sin(\pi t)
\]

\[
\left(1 + \beta(t - 1)\right) \sin^2 \left(\frac{\pi_t}{2}\right) \geq 0,
\]

then in both last functions when \( t = 1 \) they will be 0 and when \( t = 0 \) they will be 1.

6) Symmetry \( p_{0}, p_{1}, p_{2} \) and \( p_{2}, p_{1}, p_{0} \) define the same trignometric curve in different parameterizations, i.e.,

\[
r(t; \alpha, \beta; p_{0}, p_{1}, p_{2}) = r(1 - t; \beta; p_{2}, p_{1}, p_{0})
\]

Where \( 0 \leq t \leq 1, 0 \leq \alpha, \beta \leq 1 \). Then

1. \( B_{0,2}(t; \alpha, \beta) = B_{2,2}(1 - t; \beta, \alpha) \), because

\[
B_{2,2}(1 - t; \beta, \alpha) \sin^2 \left(\frac{\pi_t}{2}\right) = \left(1 + \beta(t - 1)\right) \sin^2 \left(\frac{\pi_t}{2}\right)
\]

\[
\left(1 + \beta(t - 1)\right) \sin^2 \left(\frac{\pi_t}{2}\right) \geq 0,
\]

and when \( \beta = 1 \)

\[
\left(1 + (t - 1)\right) \sin^2 \left(\frac{\pi_t}{2}\right) \geq 0
\]

and when \( \beta = 0 \)

\[
\left(1 + \beta(t - 1)\right) \sin^2 \left(\frac{\pi_t}{2}\right) \geq 0,
\]

then in both last functions when \( t = 1 \) they will be 0 and when \( t = 0 \) they will be 1.

2. \( B_{1,2}(t; \alpha, \beta) = B_{2,1}(1 - t; \beta, \alpha) \), because

\[
B_{2,1}(1 - t; \beta, \alpha) = \frac{1}{2}\beta + (\alpha - \beta)t + (\beta - (\alpha + \beta)t) \sin(\pi(1 - t))
\]

\[
(1 - t) \cos^2 \left(\frac{\pi_t}{2}\right) - (1 - \cos^2 \left(\frac{\pi_t}{2}\right))
\]

\[
= (1 - \beta t)\left(1 - \cos^2 \left(\frac{\pi_t}{2}\right)\right) + (1 + \beta(t - 1) \sin^2 \left(\frac{\pi_t}{2}\right))
\]

\[
= (1 - \beta t)\left(1 - \cos^2 \left(\frac{\pi_t}{2}\right)\right) + (1 + \beta(t - 1) \sin^2 \left(\frac{\pi_t}{2}\right))
\]

\[
= B_{0,2}(t; \alpha, \beta).
\]

Figure 6: The shape parameter \( \alpha, \beta \) with values between the rang [0,1].

Figure 7: The parameters by increasing \( \beta \) and fixing \( \alpha \).

Figure 8: The parameters \( \alpha, \beta \) with values outside the range [0,1].

Figure 7 Upon increasing \( \alpha \) by the same value, where \( \alpha, \beta \in [0,1], \) this increment can be interpreted as controlling the fullness of the curve. In Figure 8, the observed experiment, \( \beta \) is increased while \( \alpha \) remains fixed, illustrating how the curve's shape can be altered by manipulating the parameters \( \alpha \) and \( \beta \). Specifically, when \( \alpha \) and \( \beta \) both equal -1, the curve takes a linear form, as depicted in Figure 4. Conversely, when \( \alpha \) and \( \beta \) are both set to 0, the curve transforms into an ordinary quadratic Bezier-like curve, as seen in Figure 1. Notably, Figure 7 provides further insights, showcasing that the range of \( \alpha \) and \( \beta \) spans between [0,1], underscoring the versatility and control that these parameters offer in shaping the curve.

3. Arabic Script

Arabic script presents a unique and intricate challenge, setting it apart from other fonts like English. Consisting of 28 fundamental letters, written from right to left, the Arabic alphabet relies on multiple segments that closely relate to the letter's shape. Notably, most letters directly connect to the succeeding letter, imparting an overall cursive look to written text. In generating Arabic script, two key factors play a crucial role: the careful positioning of control points and the precise determination of segment numbers. These elements hold utmost significance in achieving the graceful and flowing appearance characteristic of Arabic writing [8]. Drawing the script requires the
control points for all segments that determine the shape of the letter. To adjust the shape of the letter must change the inner points for all segments.

4. Methodology
The process of creating fonts involves several important steps:

1) Firstly, paper charts are used to outline the design of the fonts. On these charts, relevant points are designated with specific coordinate levels.
2) Next, straight lines are drawn between these points, forming the initial structure of the curves.
3) If the letter's shape is not yet satisfactory, further adjustments are made by modifying the points until the desired form is achieved. This iterative process ensures the creation of appropriately shaped letters in the font.

Now apply these steps on the quadratic trigonometric Bezier-like curve with the appropriate programming language, and this study uses the MATHEMATICA software.

5. Design Arabic Font

In the forthcoming sections, the process of designing an Arabic font using MATHEMATICA software will be illustrated. To achieve the desired modifications in the curves, two key methods will be employed: first, by adding and adjusting control points, and second, by manipulating the shape parameters. These actions allow us to finely tune the curves, ensuring precise control over the font's appearance and achieving the desired artistic outcome.

Example 1

The MATHEMATICA software will be employed to design the outline of a flower using a quadratic trigonometric Bezier-like curve. This curve is characterized by a combination of quadratic and linear polynomial basis functions (2). With the aid of MATHEMATICA, we can seamlessly integrate these functions to create an aesthetically pleasing and intricate representation of the flower's outer shape.

Example 2:

In this font design example, the representation of the Arabic word "الله" (Allah) will be illustrated using a quadratic trigonometric Bezier-like curve governed by equations (1). Though MATHEMATICA software will be relied on for this demonstration, our goal is to approximate the drawing of the Arabic word "الله" to resemble the original one shown in Figure 11. This creative endeavor allows us to craft a visually captivating and accurate rendition of the word, paying homage to its cultural and linguistic significance.

Figure. 10: The original Arabic Word (الله).

Figure. 11: The quadratic, trigonometric Bezier-like curve with different shape parameters.
creating complex geometric shapes, these curves empower designers to achieve visually appealing and sophisticated results. Their widespread applications in various domains underscore their significance as indispensable tools in the field of computer graphics and design.

6. The discussion

The examples had illustrated the potency of quadratic trigonometric Bezier-like curves, effectively merging quadratic and linear polynomial basis functions to allow precise curve manipulation. In the initial instance, we had showcased the curve's adaptability by altering control points and shape parameters, revealing its capacity to achieve diverse configurations. The second example had demonstrated the curve's effectiveness in faithfully representing intricate Arabic fonts while maintaining flexibility and precision through control points and shape parameter adjustments. These results had affirmed that quadratic trigonometric Bezier-like curves are indispensable tools, providing fine-tuned control for creating aesthetically appealing designs across various applications like font design and geometric modelling.

Conclusion

Our primary focus in this study is to devise new functions using the quadratic trigonometric Bezier-like curve, with the aim of designing the Arabic word “الله” (Allah) and creating an outface of a flower. We have meticulously demonstrated and validated all properties of these new functions, manually verifying their geometric coherence through the use of MATHEMATICA software. The results of our study reveal that fonts that closely resemble the original ones can be achieved by leveraging the quadratic trigonometric Bezier-like curve. Furthermore, by employing different Bezier-like curves and diverse functions of Bezier-like functions, along with varying control points and shape parameters, the same font with intriguing variations in shape and appearance can be obtained. The flexibility of this approach is evident, as the middle control point is not specifically defined. This allows for the generation of diverse curves with the same start and end points by merely adjusting the shape parameters. Our findings indicate that all derived functions yield visually pleasing and aesthetically satisfying shapes. The successful application of quadratic trigonometric Bezier-like curves in designing fonts and flower outfaces underscores their efficacy as versatile and efficient tools in the realm of computer graphics and geometric modeling. The comprehensive analysis, along with the manual and geometric validation, reinforces the significance of these newly derived functions, offering valuable insights into their creative potential and broad applications.

References
