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Application of the Sumudu Variational Iteration Method with Atangana-Baleanu-Caputo Operator for Solving Fractional-Order Heat-Like Equations with Initial Conditions

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A B S T R A C T

Fractional calculus techniques are widely utilized across various engineering disciplines and applied sciences. Among these techniques is the Sumudu Variational Iteration Method (SVIM), which has not yet been tested with the Atangana-Baleanu-Caputo fractional derivative in academic literature. This work aims to explore the application of SVIM for solving fractional-order partial differential equations using the Atangana-Baleanu-Caputo derivative. The method integrates the Sumudu transform with the variational iteration method. To demonstrate the effectiveness and validity of SVIM, we apply it to solve one-dimensional (1-D), two-dimensional (2-D), and three-dimensional (3-D) fractional-order heat-like partial differential equations. The results indicate that SVIM is both convergent and efficient for solving these types of fractional partial differential equations.

تطبيق طريقة تحليل سمودو التكرار المتغير لحل معادلات شبيهة الحرارة ذات الرتب الكسرية مع المؤثر الكسري كابوتو- أتانجانا-باليانو

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الكلمات المفتاحية:
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طريقة التكرار المتغير

الملخص

تُستخدم تقنيات حساب التفاضل والتكامل الكسري بشكل فعال في العديد من مختلف التخصصات الهندسية والعلوم التطبيقية. ومن بين التقنيات التي تم أخذها بعين الاعتبار طريقة تحليل سمودو التكرار المتغير، والتي لم يختبرها الأكاديميون باستخدام المؤثر الكسري لكابوتو- أتانجانا—باليابو. ويهدف العمل الحالي إلى دراسة طريقة تحليل سمودو التكرار المتغير لحل معادلات تفاضلية ذات الرتب الكسرية. وهذه الطريقة تجمع بين تحويل سمودو وطريقة التكرار المتغير. ولإثبات فعالية وصلاحية الطريقة، لقد طبقت لحل المعادلات التفاضلية الجزئية ذات الرتب الكسرية الشبيهة الحرارة. وأخيرًا، الطريقة تكون متقاربة وفعالة لحل مثل هذا النوع من المعادلات التفاضلية الجزئية ذات الرتب الكسرية.

1.Introduction

Fractional calculus has recently gained prominence in various fields, including engineering, physics, applied mathematics, and applied sciences [1, 2]. Both local and nonlocal definitions of fractional derivatives have been explored, with nonlocal derivatives often proving to be more intriguing. Among these, the Riemann-Liouville and Caputo definitions are widely discussed in the literature [3-5]. More contemporary definitions, such as the Caputo-Fabrizio and Atangana-Baleanu derivatives, have also emerged [6–8]. Researchers have shown considerable interest in solving fractional heat- and wave-like equations with variable coefficients [9–16].

The standard variational iteration method (VIM), initially proposed by He [17], has been a traditional approach in this context. However, its implementation in fractional differential equations is often slow to converge due to the direct use of Lagrange multipliers [18]. The Sumudu Variational Iteration Method (SVIM) combines the Sumudu transform with the variational iteration method, offering a potentially more efficient approach.

This study aims to apply SVIM to solve various forms of three-

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dimensional fractional-order heat-like partial differential equations involving the Caputo-Atangana-Baleanu derivatives. Recent advancements in numerical methods include the HPSTM, which has been used to solve the fractional Klein-Gordon equation [19], the double Sumudu transform [20], the VHPM applied to the generalized time-space fractional Schrödinger equation [21], and the HPM used for quadratic Riccati differential equations of fractional order [22], as well as the SDM for fractional differential equations [23, 24]. This paper extends these methods to fractional-order partial differential equations and provides illustrative examples of solving fractional-order heat-like partial differential equations.

2. Preliminary" to illustrate the idea's significant concepts, and definition

We start with the definitions of the Atangana–Baleanu fractional derivative and the Sumudu transform, which will be used further in this work.

Definition 1: [7,25] The fractional derivative with the Atangana-Baleanu-Caputo operator, which is defined as:

$$ {}^{ABC}_{0}D^{\mu}_{\tau}\,\vartheta(\tau)=\frac{\delta(\mu)}{(1-\mu)}\int_0^{\tau}E_{\mu}\left[-\frac{\mu(\tau-\upsilon)^{\mu}}{(1-\mu)}\right]\vartheta'(\upsilon)\,d\upsilon, \tag{1}$$

where ${}^{ABC}D^{\mu}_{t}$ is the Atangana-Baleanu-Caputo operator, $\vartheta'(\upsilon)$ is the derivative of ϑ', $\delta(\mu)$ is a normalization function with $\delta(0)=1, \delta(1)=1$ and E_{μ} is Mittag-Leffer function.

Definition 2: [26,27] The Sumudu transform is defined over a group of functions B.

$$B=\left\{\begin{matrix}\vartheta(\tau)/\exists\,\mathbb{N},\,t_1,\,t_2>0,\,\left|\vartheta(\tau)\right|<N\,\exp\left(\dfrac{|\tau|}{t_i}\right),\\ if\ \tau\in(-1)^i\times\left[0,\infty\right)\end{matrix}\right\}. \tag{2}$$

By the following integral

$$S\left[\vartheta(\tau)\right](\upsilon)=\int_0^{\infty}\mathrm{e}^{-\tau}\,\vartheta(\tau\upsilon)\ d\tau,\quad \tau>0, \tag{3}$$

where υ is a Sumudu transform parameter.

Some special properties of the Sumudu transform are as follows:

$S\left[1\right]=1,$

$S\left[\dfrac{\tau^{m}}{\Gamma\left(m+1\right)}\right]=\upsilon^{m}.$

Definition 3: [25] The sumudu transform for the Atangana-Baleanu-Caputo fractional operator for $0<\mu\le 1$ is defined as:

$$S\left[{}^{ABC}_{0}D^{\mu}_{t}\,\vartheta(\tau)\right](\upsilon)=\frac{\delta(\mu)}{1-\mu+\mu\upsilon^{\mu}}\left(S\left[\upsilon(\tau)\right]-\vartheta(0)\right), \tag{4}$$

where $\delta(\mu)$ is a normalization function with $\delta(0)=1, \delta(1)=1$.

3. Analysis of Sumudu variational iteration method (SVIM), uniqueness and convergence

3.1 Analysis of Sumudu variational iteration method (SVIM)

In Atangana-Baleanu-Caputo operator sense, consider the following, a general nonlinear partial differential equation with initial condition:

$$ {}^{ABC}_{0}D^{\mu}_{\tau}\,\vartheta(x,\tau)+L\,\vartheta(x,\tau)+N\,\vartheta(x,\tau)=f\ (x,\tau),\quad 0<\mu\le 1. \tag{5}$$

subject to the initial condition

$$\vartheta(x,0)=\vartheta_0(x), \tag{6}$$

where ${}^{ABC}_{0}D^{\mu}_{\tau}$ is the μ order fractional Atangana-Baleanu-Caputo derivative, ϑ is the unknown function, L and N are linear and nonlinear operators, and f is the source term.

Applying the Sumudu transform on (5), given by

$$S\left[{}^{ABC}_{0}D^{\mu}_{\tau}\,\vartheta(x,\tau)\right]=S\left[\,f(x,\tau)-L\,\vartheta(x,\tau)-N\vartheta(x,\tau)\right], \tag{7}$$

from Definition 3 and Eq. (6).

$$S\left[\,\vartheta(x,\tau)\right]=\vartheta(x,0)+\left(1-\mu+\mu\,\upsilon^{\mu}\right)S\left[\,f\,(x,\tau)-L\,\vartheta(x,\tau)-N\vartheta(x,\tau)\right]. \tag{8}$$

Using the inverse Sumudu transform, given by

$$\vartheta(x,\tau)=S^{-1}\left[\left[\,\vartheta(x,0)+\left(1-\mu+\mu\,\upsilon^{\mu}\right)S\left[\,f\,(x,\tau)-L\,\vartheta(x,\tau)-N\ \vartheta(x,\tau)\right]\right]\right]. \tag{9}$$

By differentiating Eq.(9) for τ,

$$\frac{\partial\,\vartheta(x,\tau)}{\partial\tau}=\frac{\partial}{\partial\tau}\left\{S^{-1}\left[\,\vartheta(x,0)+\left(1-\mu+\mu\,\upsilon^{\mu}\right)S\left[\vartheta(x,0)+\left(1-\mu+\mu\,\upsilon^{\mu}\right)S\left[\,f\,(x,\tau)-L\,\vartheta(x,\tau)-N\vartheta(x,\tau)\right]\right]\right]\right\}. \tag{10}$$

According to the variational iteration method [23], it can construct a correct functional for Eq. (10) as follows:

$$\vartheta_{n+1}(x,\tau)=\vartheta_n(x,\tau)+\int_0^{\tau}\lambda\left[\frac{\partial\vartheta_n(x,\omega)}{\partial\omega}\left[-\frac{\partial}{\partial\omega}\left\{S^{-1}\left[\vartheta(x,0)+\left(1-\mu+\mu\,\upsilon^{\mu}\right)S\left[\,f\,(x,\tau)-L\,\vartheta_n(x,\omega)-N\ \vartheta_n(x,\omega)\right]\right\}\right]\right]d\omega, \tag{11}$$

where λ is Lagrange multiplier, ϑ_0 is an initial approximation which must be chosen suitably, and $\tilde{\vartheta}_n$ is the restricted variation, that is $\delta\tilde{\vartheta}_n=0$.

Using variation theory, λ for Eq.(11) may be calculated as:

$1+\lambda\big|_{\omega=\tau}=0.$

The Lagrange multiplier, therefore, can be easily identified as:

$\lambda=-1.$

From Eq. (11), we obtain

$$\vartheta_{n+1}(x,\tau)=\vartheta_n(x,\tau)-\int_0^{\tau}\left[\frac{\partial\vartheta_n(x,\omega)}{\partial\omega}\left[-\frac{\partial}{\partial\omega}\left\{S^{-1}\left[\vartheta(x,0)+\left(1-\mu+\mu\,\upsilon^{\mu}\right)S\left[\,f\,(x,\tau)-L\,\vartheta_n(x,\omega)-N\ \vartheta_n(x,\omega)\right]\right\}\right]\right]d\omega, \tag{12}$$

where $n=0,1,2,....$

The approximate solution is given by

$$\vartheta(x,\tau)=\lim_{n\to\infty}\vartheta_n(x,\tau). \tag{13}$$

3.2 Uniqueness theorem

Let $B=K\left(\Omega,\mathbb{R}\right)$ denote the Banach space of all the continuous functions ϑ on $\Omega=\mathbb{R}\times\left[0,T\right]$ with the norm

$$norm\left\|\,\vartheta(x,\tau)\right\|=\max_{(x,\tau)\in\Omega}\left|\,\vartheta(x,\tau)\right|. \tag{14}$$

Theorem 3.1 Suppose that L and N are also Lipschitzion with $\left|L\,\vartheta-L\varpi\right|<k_1\left|\vartheta-\varpi\right|$ and $\left|N\,\vartheta-N\,\varpi\right|<k_2\left|\vartheta-\varpi\right|$ where k_1 and k_2 are Lipschitz constants. ϑ and ϖ are two different function values. Then the solution (13) is a unique solution for Eq. (5).

$$0<\left(k_1+k_2\right)\left(\left(1-\mu\right)+\frac{\mu\tau^{\mu}}{\Gamma\left(\mu+1\right)}\right)\le 1$$

Proof. At the beginning, we define the operator $g:B\to B$ where

$$\vartheta_{n+1}(x,\tau)=\vartheta_n(x,\tau)-S^{-1}\left[\left(1-\mu+\mu\,\upsilon^{\mu}\right)S\left[\,f\ (x,\tau)-L\,\vartheta_n(x,\omega)-N\ \ \vartheta_n(x,\omega)\right]\right\}\Big]\Big]. \tag{15}$$

In order to investigate the existence and uniqueness of the solution to Eq. (5), we use Banach fixed point theorem. For this, let $\vartheta,\varpi\in B$, we have

$$\|g\vartheta - g\varpi\| = \max_{(x,\tau)\in\Omega}\left|\begin{array}{l} S^{-1}\left[\left(1-\mu+\mu\upsilon^{\mu}\right)S\left[L\vartheta+N\vartheta\right]\right] \\ -S^{-1}\left[\left(1-\mu+\mu\upsilon^{\mu}\right)S\left[L\varpi+N\varpi\right]\right]\end{array}\right|,$$

$$= \max_{(x,\tau)\in\Omega}\left|\begin{array}{l} S^{-1}\left[\left(1-\mu+\mu\upsilon^{\mu}\right)S\left[L\vartheta-L\varpi\right]\right] \\ +S^{-1}\left[\left(1-\mu+\mu\upsilon^{\mu}\right)S\left[N\vartheta-N\varpi\right]\right]\end{array}\right|,$$

$$\le \max_{(x,\tau)\in\Omega}\left|\begin{array}{l} k_1 S^{-1}\left[\left(1-\mu+\mu\upsilon^{\mu}\right)S\left[|\vartheta-\varpi|\right]\right] \\ +k_2 S^{-1}\left[\left(1-\mu+\mu\upsilon^{\mu}\right)S\left[|\vartheta-\varpi|\right]\right]\end{array}\right|, \tag{16}$$

$$\le \max_{(x,\tau)\in\Omega}\left|\left(k_1+k_2\right)S^{-1}\left[\left(1-\mu+\mu\upsilon^{\mu}\right)S\left[|\vartheta-\varpi|\right]\right]\right|,$$

$$=\left(k_1+k_2\right)\left(1-\mu+\frac{\mu\tau^{\mu}}{\Gamma(\mu+1)}\right)\|\vartheta-\varpi\|.$$

g is a contraction as $0<\left(k_1+k_2\right)\left(1-\mu+\frac{\mu\tau^{\mu}}{\Gamma(\mu+1)}\right)<1.$ From the Banach fixed point theorem.

3.3 Convergence theorem

Theorem 3.2 Let $\vartheta_n(x,\tau)$ and $\vartheta(x,\tau)$ be in Banach space B. If there exists a positive constant $\sigma=\left(k_1+k_2\right)\left(1-\mu+\frac{\mu\tau^{\mu}}{\Gamma(\mu+1)}\right)\in(0,1)$ such that $\|\vartheta_{n+1}(x,\tau)\|\le\sigma\|\vartheta_n(x,\tau)\|$ for all $(x,\tau)\in\Omega$ with $\|\vartheta_1(x,\tau)-\vartheta_0(x,\tau)\|\le\infty,$ then the sequence defined by Eq. (13) with $\vartheta_0(x,\tau)=\vartheta(x,0)$ converges to $\vartheta(x,\tau),$ i.e the exact solution of Eq. (5).

Proof. To prove this theorem, it suffices to show that $\vartheta_n(x,\tau)$ is the Cauchy sequence in Banach space B.

$$\|\vartheta_n(x,\tau)-\vartheta_m(x,\tau)\| = \max_{(x,\tau)\in\Omega}\left|\vartheta_n(x,\tau)-\vartheta_m(x,\tau)\right|,$$

$$\le \max_{(x,\tau)\in\Omega}\left|\begin{array}{l} S^{-1}\left[\left(1-\mu+\mu\upsilon^{\mu}\right)S\left[L\vartheta_n(x,\tau)+N\vartheta_n(x,\tau)\right]\right] \\ -S^{-1}\left[\left(1-\mu+\mu\upsilon^{\mu}\right)S\left[L\vartheta_m(x,\tau)+N\vartheta_m(x,\tau)\right]\right]\end{array}\right|,$$

$$\le \max_{(x,\tau)\in\Omega}\left|\begin{array}{l} S^{-1}\left[\left(1-\mu+\mu\upsilon^{\mu}\right)S\left[L\vartheta_n(x,\tau)-L\vartheta_n(x,\tau)\right]\right] \\ +S^{-1}\left[\left(1-\mu+\mu\upsilon^{\mu}\right)S\left[L\vartheta_m(x,\tau)-N\vartheta_n(x,\tau)\right]\right]\end{array}\right|, \tag{17}$$

$$\le \max_{(x,\tau)\in\Omega}\left|\left(k_1+k_2\right)S^{-1}\left[\left(1-\mu+\mu\upsilon^{\mu}\right)S\left[|\vartheta_n-\vartheta_m|\right]\right]\right|,$$

$$=\left(k_1+k_2\right)\left(1-\mu+\frac{\mu\tau^{\mu}}{\Gamma(\mu+1)}\right)\|\vartheta_n-\vartheta_m\|.$$

Let $n=m+1,$ then

$$\begin{aligned}&\|\vartheta_{m+1}(x,\tau)-\vartheta_m(x,\tau)\|\le\sigma\|\vartheta_m(x,\tau)-\vartheta_{m-1}(x,\tau)\| \\ &\le\sigma^2\|\vartheta_{m-1}(x,\tau)-\vartheta_{m-2}(x,\tau)\|\le\cdots\le\sigma^m\|\vartheta_1(x,\tau)-\vartheta_0(x,\tau)\|,\end{aligned} \tag{18}$$

$$\sigma=\left(k_1+k_2\right)\left(1-\mu+\frac{\mu\tau^{\mu}}{\Gamma(\mu+1)}\right). \tag{19}$$

From the triangle inequality, we have

$$\|\vartheta_n(x,\tau)-\vartheta_m(x,\tau)\|=\left\|\begin{array}{r}\vartheta_{m+1}(x,\tau)-\vartheta_m(x,\tau)+\vartheta_{m+2}(x,\tau)-\vartheta_{m+1}(x,\tau) \\ +\cdots+\vartheta_n(x,\tau)-\vartheta_{n-1}(x,\tau)\end{array}\right\|$$

$$\begin{aligned}&\le\|\vartheta_{m+1}(x,\tau)-\vartheta_m(x,\tau)\|+\|\vartheta_{m+2}(x,\tau)-\vartheta_{m+1}(x,\tau)\| \\ &+\cdots+\|\vartheta_n(x,\tau)-\vartheta_{n-1}(x,\tau)\| \\ &\le\sigma^m\|\vartheta_1(x,\tau)-\vartheta_0(x,\tau)\|+\sigma^{m+1}\|\vartheta_1(x,\tau)-\vartheta_0(x,\tau)\| \\ &+\cdots+\sigma^{n-1}\|\vartheta_1(x,\tau)-\vartheta_0(x,\tau)\| \\ &=\sigma^m\left(1+\sigma+\cdots+\sigma^{n-m-1}\right)\|\vartheta_1(x,\tau)-\vartheta_0(x,\tau)\| \\ &\le\sigma^m\left(\frac{1-\sigma^{n-m}}{1-\sigma}\right)\|\vartheta_1(x,\tau)-\vartheta_0(x,\tau)\|.\end{aligned} \tag{20}$$

Since $0<\sigma<1,$ so $1-\sigma^{n-m}<1,$ then

$$\|\vartheta_n(x,\tau)-\vartheta_m(x,\tau)\|=\sigma^m\left(\frac{1-\sigma^{n-m}}{1-\sigma}\right)\|\vartheta_1(x,\tau)-\vartheta_0(x,\tau)\|. \tag{21}$$

But $\|\vartheta_1(x,\tau)-\vartheta_0(x,\tau)\|<\infty,$ then $\|\vartheta_n(x,\tau)-\vartheta_m(x,\tau)\|\to0$ as $n\to\infty.$ We conclude that $\{\vartheta_n(x,\tau)\}$ is a Cauchy sequence in the Banach space B. Therefore the sequence converges.

4. Applications

Example 4.1: Consider the following one-dimensional fractional heat-like equation in the Atangana-Baleanu-Caputo operator sense.

$${}^{ABC}_{\ \ 0}D^{\mu}_{\tau}\,\vartheta(x,\tau)=\frac{1}{2}x^2\frac{\partial\vartheta^2}{\partial x^2}, \tag{22}$$

whereinitial and the subject to the $0<\mu\le1,\quad 0<x\le1,\quad \tau>1,$ condition

$$\vartheta(x,0)=x^2. \tag{23}$$

The exact solution of Eq. (22) when $\mu=1$ is given:

$$\vartheta(x,\tau)=x^2e^{\tau}. \tag{24}$$

Applying the Sumudu transform on Eq. (22) and using the initial condition (23), we get

$$S\left[\vartheta(x,\tau)\right]=x^2+\left(1-\mu+\mu\nu^{\mu}\right)S\left[\frac{1}{2}x^2\frac{\partial\vartheta^2}{\partial x^2}\right], \tag{25}$$

using the inverse Sumudu transform to Eq. (25), given by

$$\vartheta(x,\tau)=x^2+S^{-1}\left[\frac{1}{2}x^2\left(1-\mu+\mu\upsilon^{\mu}\right)S\left[\frac{\partial\vartheta^2}{\partial x^2}\right]\right]. \tag{26}$$

By differentiating Eq. (26) for τ, given by

$$\frac{\partial\vartheta}{\partial\tau}=\frac{\partial}{\partial\tau}S^{-1}\left[\frac{1}{2}x^2\left(1-\mu+\mu\upsilon^{\mu}\right)S\left[\frac{\partial\vartheta^2}{\partial x^2}\right]\right], \tag{27}$$

The aforementioned technique has been used in order to create the correction functional for Eq. (27) as

$$\begin{aligned}\vartheta_{n+1}(x,\tau)=\vartheta_n-\int_0^{\tau}\lambda\Bigg[&\frac{\partial\vartheta_n}{\partial\omega} \\ &-\frac{\partial}{\partial\omega}\left\{S^{-1}\left[\frac{1}{2}x^2\left(1-\mu+\mu\upsilon^{\mu}\right)S\left[\frac{\partial\vartheta_n^{\,2}}{\partial x^2}\right]\right]\right\}\Bigg]d\omega.\end{aligned} \tag{28}$$

By variation theory, λ for Eq. (28) can be obtained as

$$1+\lambda|\ =0,\quad so,\quad \lambda=-1.$$

From Eq. (28), we get

$$\begin{aligned}\vartheta_{n+1}(x,\tau)=\vartheta_n-\int_0^{\tau}\Bigg[&\frac{\partial\vartheta_n}{\partial\omega} \\ &-\frac{\partial}{\partial\omega}\left\{S^{-1}\left[\frac{1}{2}x^2\left(1-\mu+\mu\upsilon^{\mu}\right)S\left[\frac{\partial\vartheta_n^{\,2}}{\partial x^2}\right]\right]\right\}\Bigg]d\omega.\end{aligned} \tag{29}$$

As a consequence, the approximate solution may be obtained using Eq. (12).

We start with an initial approximation

$$\vartheta_0(x,\tau)=x^2, \tag{30}$$

$$\vartheta_1(x,\tau)=\vartheta_0-\int_0^{\tau}\left[\frac{\partial\,\vartheta_0}{\partial\omega}-\frac{\partial}{\partial\omega}\left\{S^{-1}\left[\frac{1}{2}x^2\left(1-\mu+\mu\,\upsilon^{\mu}\right)S\left[\frac{\partial\vartheta_0{}^2}{\partial x^2}\right]\right]\right\}\right]d\omega, \quad (31)$$
$$=x^2\left(1+\frac{\mu\tau^{\mu}}{\Gamma(\mu+1)}\right),$$

$$\vartheta_2(x,\tau)=\vartheta_1-\int_0^{\tau}\left[\frac{\partial\vartheta_1}{\partial\omega}-\frac{\partial}{\partial\omega}\left\{S^{-1}\left[\frac{1}{2}x^2(1-\mu+\mu\,\upsilon^{\mu})\,S\left[\frac{\partial\vartheta_1{}^2}{\partial x^2}\right]\right]\right\}\right]d\omega, \quad (32)$$
$$=x^2\left(1+\frac{(2\mu-\mu^2)\,\tau^{\mu}}{\Gamma(\mu+1)}+\frac{\mu^2\tau^{2\mu}}{\Gamma(2\mu+1)}\right),$$

$$\vartheta_3(x,\tau)=x^2\left(1+\frac{\left(3\mu-3\mu^2+\mu^3\right)\tau^{\mu}}{\Gamma(\mu+1)}+\frac{\left(3\mu^2-2\mu^3\right)\tau^{2\mu}}{\Gamma(2\mu+1)}+\frac{\mu^3\tau^{3\mu}}{\Gamma(3\mu+1)}\right), \quad (33)$$

$$\vartheta_4(x,\tau)=x^2\left(1+\frac{\left(4\mu-6\mu^2+4\mu^3-\mu^4\right)\tau^{\mu}}{\Gamma(\mu+1)}+\frac{\left(6\mu^2-8\mu^3+3\mu^4\right)\tau^{2\mu}}{\Gamma(2\mu+1)}+\frac{(4\mu^3-3\mu^4)\tau^{3\mu}}{\Gamma(3\mu+1)}+\frac{\mu^4\tau^{4\mu}}{\Gamma(4\mu+1)}\right), \quad (34)$$

$$\vartheta_5(x,\tau)=x^2\left(1+\frac{\left(5\mu-10\mu^2+10\mu^3-5\mu^4+\mu^5\right)\tau^{\mu}}{\Gamma(\mu+1)}+\frac{\left(10\mu^2-20\mu^3+15\mu^4-4\mu^5\right)\tau^{2\mu}}{\Gamma(2\mu+1)}+\frac{(10\mu^3-15\mu^4+6\mu^5)\tau^{3\mu}}{\Gamma(3\mu+1)}+\frac{(5\mu^4-4\mu^5)\tau^{4\mu}}{\Gamma(4\mu+1)}+\frac{\mu^5\tau^5}{\Gamma(5\mu+1)}\right), \quad (35)$$

$$\vartheta_6(x,\tau)=x^2\left(1+\frac{\left(6\mu-15\mu^2+20\mu^3-15\mu^4+6\mu^5-\mu^6\right)\tau^{\mu}}{\Gamma(\mu+1)}+\frac{\left(15\mu^2-40\mu^3+45\mu^4-24\mu^5+5\mu^6\right)\tau^{2\mu}}{\Gamma(2\mu+1)}+\frac{(20\mu^3-45\mu^4+36\mu^5-10\mu^6)\tau^{3\mu}}{\Gamma(3\mu+1)}+\frac{(15\mu^4-24\mu^5+10\mu^6)\tau^{4\mu}}{\Gamma(4\mu+1)}+\frac{(6\mu^5-5\mu^6)\tau^{5\mu}}{\Gamma(5\mu+1)}+\frac{\mu^6\tau^{6\mu}}{\Gamma(6\mu+1)}\right), \quad (36)$$

Substituting Eq. (36) in Eq. (13), we get the 6^{th} -SVIM approximate solution of the Eq. (22).

$$\vartheta(x,\tau)=x^2\left(1+\frac{\left(6\mu-15\mu^2+20\mu^3-15\mu^4+6\mu^5-\mu^6\right)\tau^{\mu}}{\Gamma(\mu+1)}+\frac{\left(15\mu^2-40\mu^3+45\mu^4-24\mu^5+5\mu^6\right)\tau^{2\mu}}{\Gamma(2\mu+1)}+\frac{(20\mu^3-45\mu^4+36\mu^5-10\mu^6)\tau^{3\mu}}{\Gamma(3\mu+1)}+\frac{(15\mu^4-24\mu^5+10\mu^6)\tau^{4\mu}}{\Gamma(4\mu+1)}+\frac{(6\mu^5-5\mu^6)\tau^{5\mu}}{\Gamma(5\mu+1)}+\frac{\mu^6\tau^{6\mu}}{\Gamma(6\mu+1)}\right). \quad (37)$$

This is consistent with the results found in previous studies[9,10,13]

Example 4.2: Consider the following two-dimensional fractional heat-like equation in the Atangana-Baleanu-Caputo operator sense.

$${}^{ABC}_{\ 0}D^{\mu}_{\tau}\,\vartheta(x,y,\tau)=\frac{1}{2}\left(\frac{\partial^2\vartheta}{\partial x^2}+\frac{\partial^2\vartheta}{\partial y^2}\right), \quad (38)$$

Where $0<\mu\le 1,\ \ 0<x,y\le\frac{\pi}{2},\ \ \tau>0,$ and subject to the initial condition

$$\vartheta(x,y,0)=\sin x\sin y. \quad (39)$$

The exact solution of Eq. (38) when $\mu=1$ is given:

$$\vartheta(x,y,\tau)=\sin x\sin y\,e^{-\tau}. \quad (40)$$

Applying the Sumudu transform on Eq. (38) and using the initial condition Eq. (39), we get

$$S\left[\vartheta(x,y,\tau)\right]=\sin x\sin y+\left(1-\mu+\mu\,\upsilon^{\mu}\right)S\left[\frac{1}{2}\left(\frac{\partial^2\vartheta}{\partial x^2}+\frac{\partial^2\vartheta}{\partial y^2}\right)\right], \quad (41)$$

using the inverse Sumudu transform to Eq. (41), given by

$$\vartheta(x,y,\tau)=\sin x\sin y+S^{-1}\left[\frac{1}{2}\left(1-\mu+\mu\,\upsilon^{\mu}\right)S\left[\frac{\partial^2\vartheta}{\partial x^2}+\frac{\partial^2\vartheta}{\partial y^2}\right]\right]. \quad (42)$$

By differentiating Eq. (42) for τ , given by

$$\frac{\partial\vartheta}{\partial t}=\frac{\partial}{\partial\tau}S^{-1}\left[\frac{1}{2}\left(1-\mu+\mu\,v^{\mu}\right)S\left[\frac{\partial^2\vartheta}{\partial x^2}+\frac{\partial^2\vartheta}{\partial y^2}\right]\right], \quad (43)$$

The aforementioned technique has been used in order to create the correction functional for Eq. (43) as

$$\vartheta_{n+1}(x,y,\tau)=\vartheta_n-\lambda\int_0^{\tau}\left[\frac{\partial\,\vartheta_n}{\partial\omega}-\frac{\partial}{\partial\omega}\left\{S^{-1}\left[\frac{1}{2}\left(1-\mu+\mu\,\upsilon^{\mu}\right)S\left[\frac{\partial\vartheta_n{}^2}{\partial x^2}+\frac{\partial^2\vartheta_n}{\partial y^2}\right]\right]\right\}\right]d\omega. \quad (44)$$

By variation theory, λ for Eq. (44) can be obtained as

$$1+\lambda|=0,\quad so,\quad \lambda=-1.$$

From Eq. (44), we get

$$\vartheta_{n+1}(x,y,\tau)=\vartheta_n-\int_0^{\tau}\left[\frac{\partial\,\vartheta_n}{\partial\omega}-\frac{\partial}{\partial\omega}\left\{S^{-1}\left[\frac{1}{2}\left(1-\mu+\mu\,\upsilon^{\mu}\right)S\left[\frac{\partial\vartheta_n{}^2}{\partial x^2}+\frac{\partial^2\vartheta_n}{\partial y^2}\right]\right]\right\}\right]d\omega. \quad (45)$$

As a consequence, the approximate solution may be obtained using Eq. (12).

We start with an initial approximation

$$\vartheta_0(x,y,\tau)=\sin x\sin y, \quad (46)$$

$$\vartheta_1(x,y,\tau)=\vartheta_0-\int_0^{\tau}\left[\frac{\partial\,\vartheta_n}{\partial\omega}-\frac{\partial}{\partial\omega}\left\{S^{-1}\left[\frac{1}{2}\left(1-\mu+\mu\,\upsilon^{\mu}\right)S\left[\frac{\partial\vartheta_0^{\,2}}{\partial x^2}+\frac{\partial^2\vartheta_0}{\partial y^2}\right]\right]\right\}\right]d\omega. \quad (47)$$

$$=\sin x\sin y\left(1-\frac{\mu\tau^{\mu}}{\Gamma(\mu+1)}\right),$$

$$\vartheta_2(x,y,\tau)=\vartheta_1-\int_0^{\tau}\left[\frac{\partial\,\vartheta_n}{\partial\omega}-\frac{\partial}{\partial\omega}\left\{S^{-1}\left[\frac{1}{2}\left(1-\mu+\mu\,\upsilon^{\mu}\right)S\left[\frac{\partial\vartheta_1^{\,2}}{\partial x^2}+\frac{\partial^2\vartheta_1}{\partial y^2}\right]\right]\right\}\right]d\omega. \quad (48)$$

$$=\sin x\sin y\left(1-\frac{\left(2\mu-\mu^2\right)\tau^{\mu}}{\Gamma(\mu+1)}+\frac{\mu^2\tau^{2\mu}}{\Gamma(2\mu+1)}\right),$$

$$\vartheta_3(x,y,\tau)=\sin x\sin y\left(1-\frac{\left(3\mu-3\mu^2+\mu^3\right)\tau^{\mu}}{\Gamma(\mu+1)}+\frac{\left(3\mu^2-2\mu^3\right)\tau^{2\mu}}{\Gamma(2\mu+1)}-\frac{\mu^3\tau^{3\mu}}{\Gamma(3\mu+1)}\right), \quad (49)$$

$$\vartheta_4(x,y,\tau)=\sin x\,\sin y\left(1-\frac{\left(4\mu-6\mu^2+4\mu^3-\mu^4\right)\tau^{\mu}}{\Gamma(\mu+1)}+\frac{\left(6\mu^2-8\mu^3+3\mu^4\right)\tau^{2\mu}}{\Gamma(2\mu+1)}-\frac{(4\mu^3-3\mu^4)\tau^{3\mu}}{\Gamma(3\mu+1)}+\frac{\mu^4\tau^{4\mu}}{\Gamma(4\mu+1)}\right), \quad (50)$$

$$\vartheta_5(x,y,\tau)=\sin x\,\sin y\left(1-\frac{\left(5\mu-10\mu^2+10\mu^3-5\mu^4+\mu^5\right)\tau^{\mu}}{\Gamma(\mu+1)}+\frac{\left(10\mu^2-20\mu^3+15\mu^4-4\mu^5\right)\tau^{2\mu}}{\Gamma(2\mu+1)}-\frac{(10\mu^3-15\mu^4+6\mu^5)\tau^{3\mu}}{\Gamma(3\mu+1)}+\frac{(5\mu^4-4\mu^5)\tau^{4\mu}}{\Gamma(4\mu+1)}-\frac{\mu^5\tau^5}{\Gamma(5\mu+1)}\right), \quad (51)$$

$$\vartheta_6(x,y,\tau)=\sin x\,\sin y\left(1-\frac{\left(6\mu-15\mu^2+20\mu^3-15\mu^4+6\mu^5-\mu^6\right)\tau^{\mu}}{\Gamma(\mu+1)}+\frac{\left(15\mu^2-40\mu^3+45\mu^4-24\mu^5+5\mu^6\right)\tau^{2\mu}}{\Gamma(2\mu+1)}-\frac{(20\mu^3-45\mu^4+36\mu^5-10\mu^6)\tau^{3\mu}}{\Gamma(3\mu+1)}+\frac{(15\mu^4-24\mu^5+10\mu^6)\tau^{4\mu}}{\Gamma(4\mu+1)}-\frac{(6\mu^5-5\mu^6)\tau^{5\mu}}{\Gamma(5\mu+1)}+\frac{\mu^6\tau^{6\mu}}{\Gamma(6\mu+1)}\right), \quad (52)$$

Substituting Eq. (52) in Eq. (13), we get the 6^{th} -SVIM approximate solution of the Eq. (38).

$$\vartheta(x,y,\tau)=\sin x\,\sin y\left(1-\frac{\left(6\mu-15\mu^2+20\mu^3-15\mu^4+6\mu^5-\mu^6\right)\tau^{\mu}}{\Gamma(\mu+1)}+\frac{\left(15\mu^2-40\mu^3+45\mu^4-24\mu^5+5\mu^6\right)\tau^{2\mu}}{\Gamma(2\mu+1)}-\frac{(20\mu^3-45\mu^4+36\mu^5-10\mu^6)\tau^{3\mu}}{\Gamma(3\mu+1)}+\frac{(15\mu^4-24\mu^5+10\mu^6)\tau^{4\mu}}{\Gamma(4\mu+1)}-\frac{(6\mu^5-5\mu^6)\tau^{5\mu}}{\Gamma(5\mu+1)}+\frac{\mu^6\tau^{6\mu}}{\Gamma(6\mu+1)}\right), \quad (53)$$

This is consistent with the results found in previous studies[9,10,13]

Example 4.3: Consider the following three-dimensional fractional heat-like equation in the Atangana-Baleanu-Caputo operator sense.

$${}^{ABC}_{\ 0}D^{\mu}_{\tau}\,\vartheta(x,y,z,t)=x^4y^4z^4+\frac{1}{36}\left(x^2\frac{\partial^2\vartheta}{\partial x^2}+y^2\frac{\partial^2\vartheta}{\partial y^2}+z^2\frac{\partial^2\vartheta}{\partial z^2}\right), \quad (54)$$

where $0<\mu\le 1,\quad 0<x,y,z\le 1,\quad \tau>0,$ and subject to the initial condition

$$\vartheta(x,y,z,0)=0. \quad (55)$$

The exact solution of Eq. (54) when $\mu=1$ is given:

$$\vartheta(x,y,z,\tau)=x^4y^4z^4\left(e^{\tau}-1\right). \quad (56)$$

Applying the Sumudu transform on (54) and using the initial condition (55), we get

$$S\left[\vartheta(x,y,z,\tau)\right]=\left(1-\mu+\mu\,\upsilon^{\mu}\right)S\left[x^4y^4z^4+\frac{1}{36}\left(x^2\frac{\partial^2\vartheta}{\partial x^2}+y^2\frac{\partial^2\vartheta}{\partial y^2}+z^2\frac{\partial^2\vartheta}{\partial z^2}\right)\right], \quad (57)$$

using the inverse Sumudu transform to (57), given by

$$\vartheta(x,y,z,\tau)=S^{-1}\left[\left(1-\mu+\mu\,\upsilon^{\mu}\right)S\left[x^4y^4z^4+\frac{1}{36}\left(x^2\frac{\partial^2\vartheta}{\partial x^2}+y^2\frac{\partial^2\vartheta}{\partial y^2}+z^2\frac{\partial^2\vartheta}{\partial z^2}\right)\right]\right]. \quad (58)$$

By differentiating Eq. (58) for τ, given by

$$\frac{\partial\vartheta}{\partial\tau}=\frac{\partial}{\partial\tau}S^{-1}\left[\left(1-\mu+\mu\,\upsilon^{\mu}\right)S\left[x^4y^4z^4+\frac{1}{36}\left(x^2\frac{\partial^2\vartheta}{\partial x^2}+y^2\frac{\partial^2\vartheta}{\partial y^2}+z^2\frac{\partial^2\vartheta}{\partial z^2}\right)\right]\right], \quad (59)$$

The aforementioned technique has been used in order to create the correction functional for Eq. (59) as

$$\vartheta_{n+1}(x,y,z,\tau)=\vartheta_n-\lambda\int_0^{\tau}\left[\frac{\partial\,\vartheta_n}{\partial\omega}-\frac{\partial}{\partial\omega}\left\{S^{-1}\left[\left(1-\mu+\mu\,\upsilon^{\mu}\right)S\left[x^4y^4z^4+\frac{1}{36}\left(x^2\frac{\partial^2\vartheta_n}{\partial x^2}+y^2\frac{\partial^2\vartheta_n}{\partial y^2}+z^2\frac{\partial^2\vartheta_n}{\partial z^2}\right)\right]\right]\right\}\right]d\omega. \quad (60)$$

By variation theory, λ for Eq. (60) can be obtained as

$1+\lambda|=0,\quad so,\quad \lambda=-1.$

From Eq. (60), we get

$$\vartheta_{n+1}(x,y,z,\tau)=\vartheta_n-\int_0^\tau\left[\frac{\partial \vartheta_n}{\partial\omega}-\frac{\partial}{\partial\omega}\left\{S^{-1}\left[\left(1-\mu+\mu\,\upsilon^\mu\right)S\left[x^4y^4z^4+\frac{1}{36}\left(x^2\frac{\partial^2\vartheta_n}{\partial x^2}+y^2\frac{\partial^2\vartheta_n}{\partial y^2}+z^2\frac{\partial^2\vartheta_n}{\partial z^2}\right)\right]\right]\right\}\right]d\omega. \tag{61}$$

We start with an initial approximation

$$\vartheta_0(x,y,z,0)=0, \tag{62}$$

$$\vartheta_1(x,y,z,\tau)=\vartheta_0-\int_0^\tau\left[\frac{\partial \vartheta_0}{\partial\omega}-\frac{\partial}{\partial\omega}\left\{S^{-1}\left[\left(1-\mu+\mu\,\upsilon^\mu\right)S\left[x^4y^4z^4+\frac{1}{36}\left(x^2\frac{\partial^2\vartheta_0}{\partial x^2}+y^2\frac{\partial^2\vartheta_0}{\partial y^2}+z^2\frac{\partial^2\vartheta_0}{\partial z^2}\right)\right]\right]\right\}\right]d\omega. \tag{63}$$

$$=x^4y^4z^4\left(\frac{\mu\tau^\mu}{\Gamma(\mu+1)}\right),$$

$$\vartheta_2(x,y,z,\tau)=\vartheta_1-\int_0^\tau\left[\frac{\partial \vartheta_1}{\partial\omega}-\frac{\partial}{\partial\omega}\left\{S^{-1}\left[\left(1-\mu+\mu\,\upsilon^\mu\right)S\left[x^4y^4z^4+\frac{1}{36}\left(x^2\frac{\partial^2\vartheta_1}{\partial x^2}+y^2\frac{\partial^2\vartheta_1}{\partial y^2}+z^2\frac{\partial^2\vartheta_1}{\partial z^2}\right)\right]\right]\right\}\right]d\omega \tag{64}$$

$$=x^4y^4z^4\left(\frac{\left(2\mu-\mu^2\right)\tau^\mu}{\Gamma(\mu+1)}+\frac{\mu^2\tau^{2\mu}}{\Gamma(2\mu+1)}\right),$$

$$\vartheta_3(x,y,z,\tau)=x^4y^4z^4\left(\frac{\left(3\mu-3\mu^2+\mu^3\right)\tau^\mu}{\Gamma(\mu+1)}+\frac{\left(3\mu^2-2\mu^3\right)\tau^{2\mu}}{\Gamma(2\mu+1)}+\frac{\mu^3\tau^{3\mu}}{\Gamma(3\mu+1)}\right), \tag{65}$$

$$\vartheta_4(x,y,z,\tau)=x^4y^4z^4\left(\frac{\left(4\mu-6\mu^2+4\mu^3-\mu^4\right)\tau^\mu}{\Gamma(\mu+1)}+\frac{\left(6\mu^2-8\mu^3+3\mu^4\right)\tau^{2\mu}}{\Gamma(2\mu+1)}+\frac{(4\mu^3-3\mu^4)\tau^{3\mu}}{\Gamma(3\mu+1)}+\frac{\mu^4\tau^{4\mu}}{\Gamma(4\mu+1)}\right), \tag{66}$$

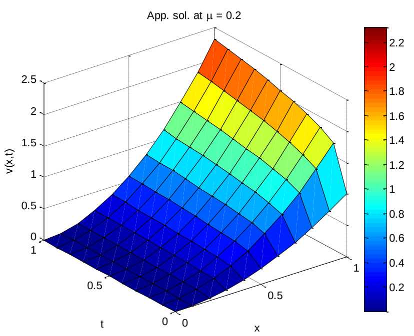
$$\vartheta_5(x,y,z,\tau)=x^4y^4z^4\left(\frac{\left(5\mu-10\mu^2+10\mu^3-5\mu^4+\mu^5\right)\tau^\mu}{\Gamma(\mu+1)}+\frac{\left(10\mu^2-20\mu^3+15\mu^4-4\mu^5\right)\tau^{2\mu}}{\Gamma(2\mu+1)}+\frac{(10\mu^3-15\mu^4+6\mu^5)\tau^{3\mu}}{\Gamma(3\mu+1)}+\frac{(5\mu^4-4\mu^5)\tau^{4\mu}}{\Gamma(4\mu+1)}+\frac{\mu^5\tau^{5\mu}}{\Gamma(5\mu+1)}\right), \tag{67}$$

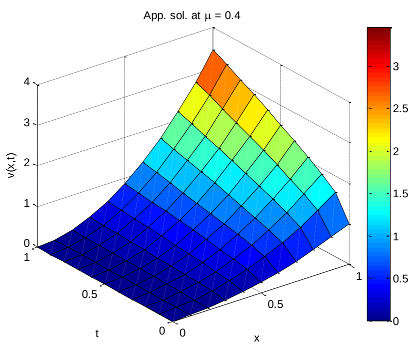
$$\vartheta_6(x,y,z,\tau)=x^4y^4z^4\left(\frac{\left(6\mu-15\mu^2+20\mu^3-15\mu^4+6\mu^5-\mu^6\right)\tau^\mu}{\Gamma(\mu+1)}+\frac{\left(15\mu^2-40\mu^3+45\mu^4-24\mu^5+5\mu^6\right)\tau^{2\mu}}{\Gamma(2\mu+1)}+\frac{(20\mu^3-45\mu^4+36\mu^5-10\mu^6)\tau^{3\mu}}{\Gamma(3\mu+1)}+\frac{(15\mu^4-24\mu^5+10\mu^6)\tau^{4\mu}}{\Gamma(4\mu+1)}+\frac{(6\mu^5-5\mu^6)\tau^{5\mu}}{\Gamma(5\mu+1)}+\frac{\mu^6\tau^{6\mu}}{\Gamma(6\mu+1)}\right), \tag{68}$$

Substituting Eq. (68) in Eq. (13), we get the 6^{th} -SVIM approximate solution of the Eq. (54).

$$\vartheta(x,y,z,\tau)=x^4y^4z^4\left(\frac{\left(6\mu-15\mu^2+20\mu^3-15\mu^4+6\mu^5-\mu^6\right)\tau^\mu}{\Gamma(\mu+1)}+\frac{\left(15\mu^2-40\mu^3+45\mu^4-24\mu^5+5\mu^6\right)\tau^{2\mu}}{\Gamma(2\mu+1)}+\frac{(20\mu^3-45\mu^4+36\mu^5-10\mu^6)\tau^{3\mu}}{\Gamma(3\mu+1)}+\frac{(15\mu^4-24\mu^5+10\mu^6)\tau^{4\mu}}{\Gamma(4\mu+1)}+\frac{(6\mu^5-5\mu^6)\tau^{5\mu}}{\Gamma(5\mu+1)}+\frac{\mu^6\tau^{6\mu}}{\Gamma(6\mu+1)}\right). \tag{69}$$

This is consistent with the results found in previous studies[10].





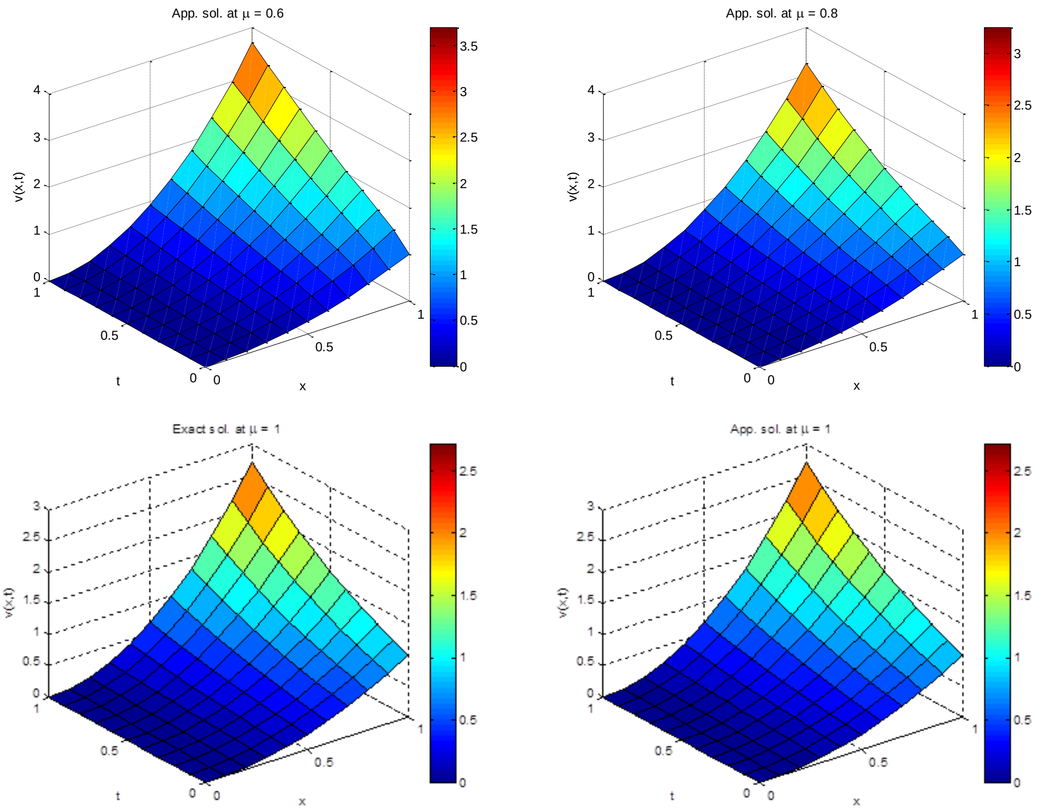


Figure 1. 3D-Surface behavior of the 6^{th} -SVIM and the exact solution for Example 1 when $\tau,\ x \in [0,1]$, for diverse of $\mu = 0.2, 0.4, 0.6, 0.8, 1.$

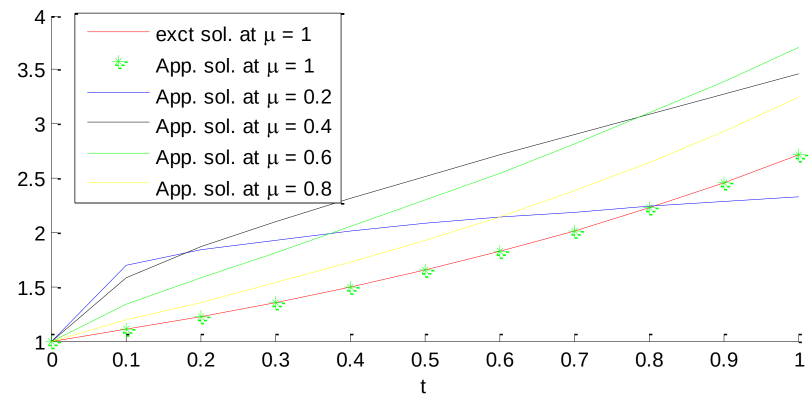
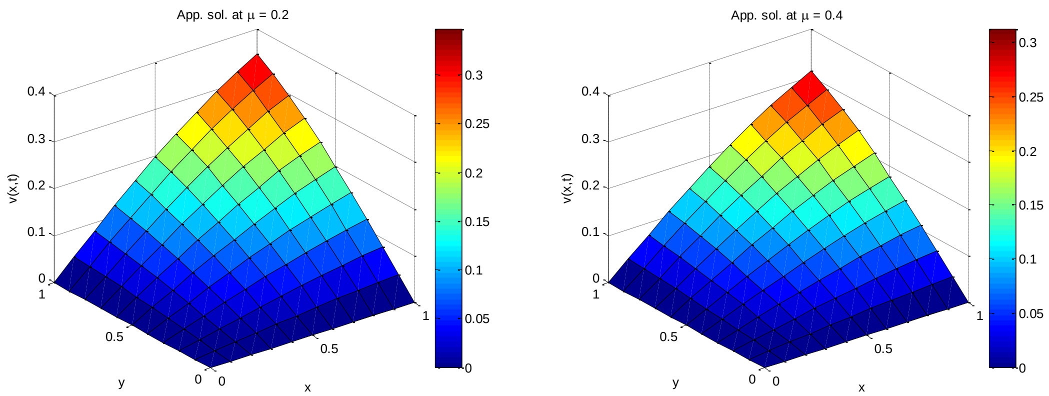


Figure 2. 2D-Surface behavior of the 6^{th} -SVIM and the exact solution for Example 1 when $\tau \in [0,1]$ and $x = 1$ for diverse of $\mu = 0.2, 0.4, 0.6, 0.8, 1.$



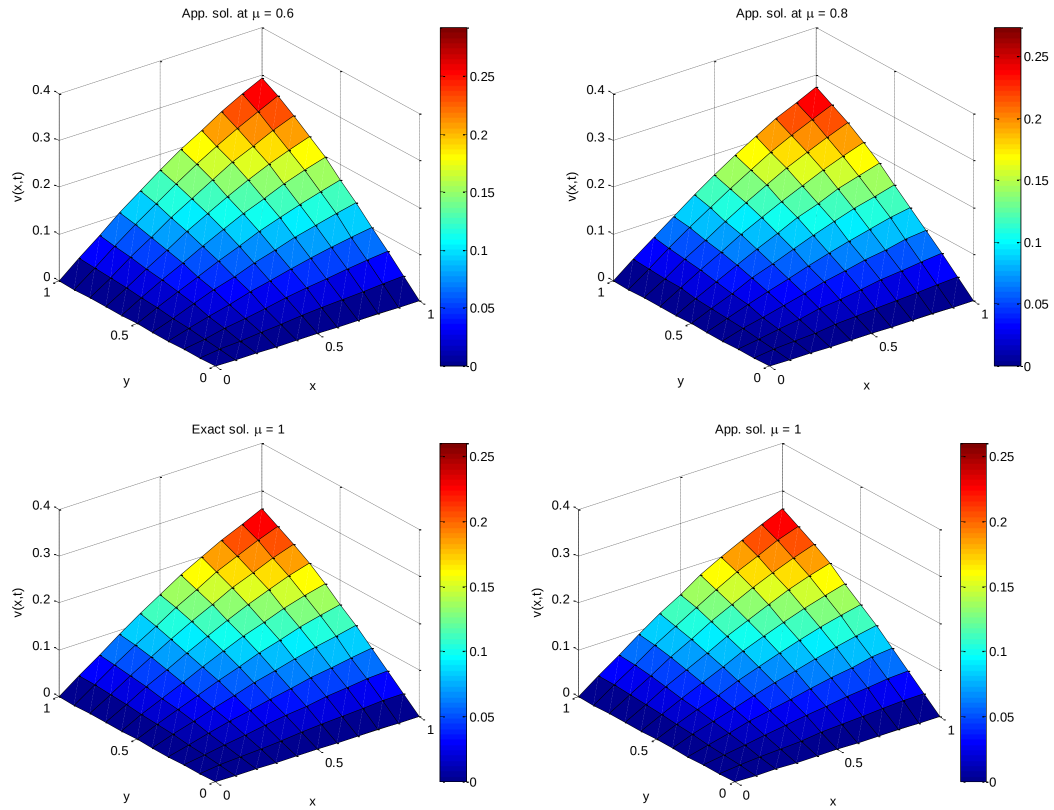


Figure 3. 3D-Surface behavior of the 6^{th} -SVIM and the exact solution for Example 2 when $x, y \in [0,1]$ and $t = 1$ for diverse of $\mu = 0.2, 0.4, 0.6, 0.8, 1.$

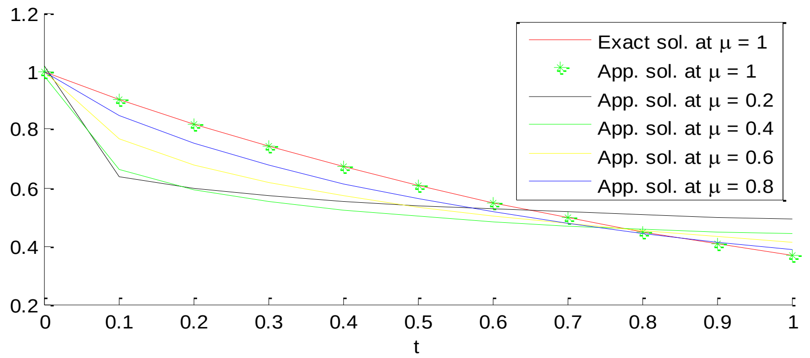
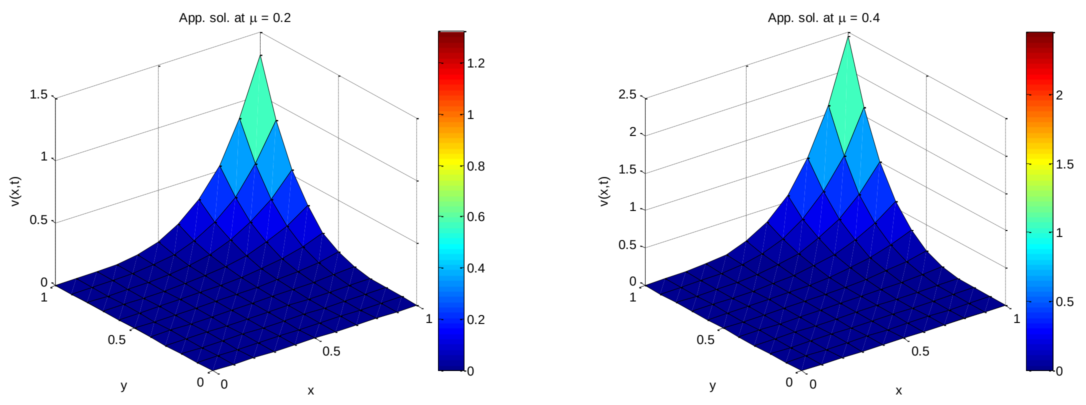


Figure 4. 2D-Surface behavior of the 6^{th} -SVIM and the exact solution for Example 2 when $\tau \in [0,1]$ and $x = y = 1$ for diverse of $\mu = 0.2, 0.4, 0.6, 0.8, 1.$



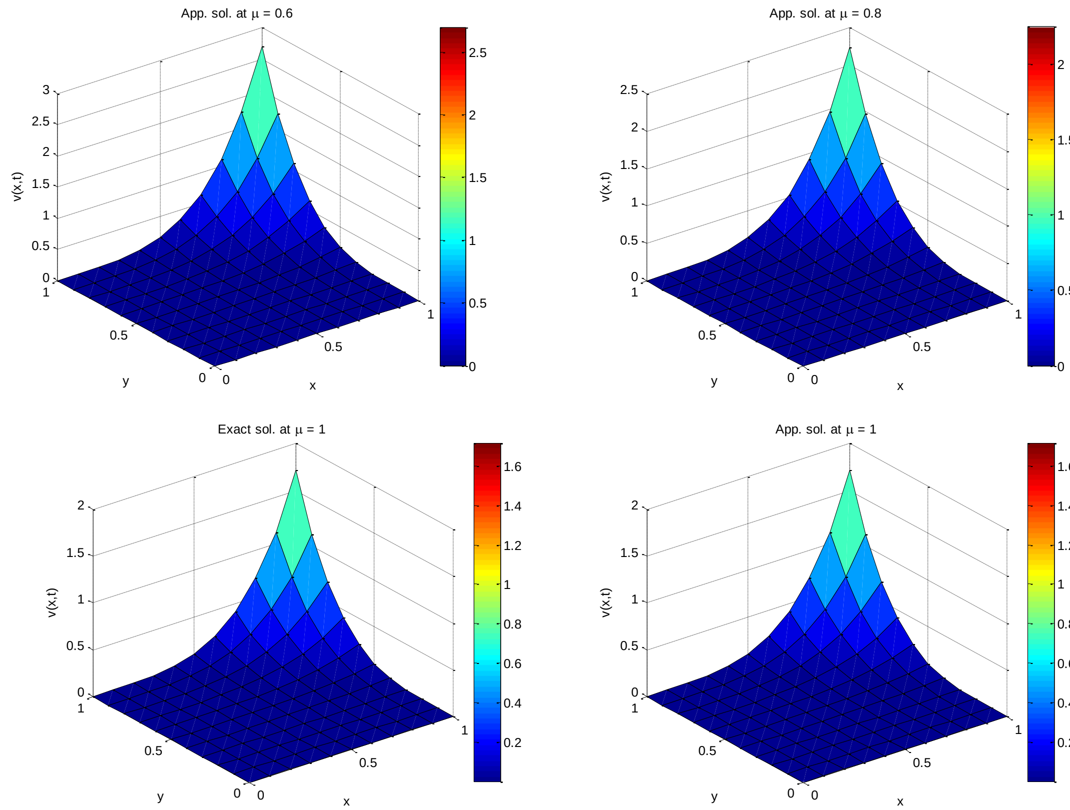


Figure 5. 3D-Surface behavior of the 6^{th} -SVIM and the exact solution for Example 3 when $x, y \in [0,1]$ and $t = z = 1$ for diverse of $\mu = 0.2, 0.4, 0.6, 0.8, 1.$

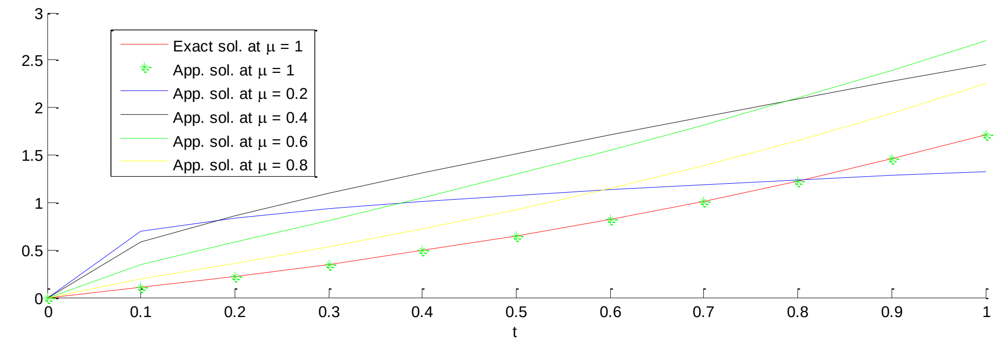


Figure 6. 2D-Surface behavior of the 6^{th} -SVIM and the exact solution for Example 3 when $\tau \in [0,1]$ and $x = y = z = 1$ for diverse of $\mu = 0.2, 0.4, 0.6, 0.8, 1.$

Table 1. Numerical results of the 6^{th} -SVIM and exact solutions for $\mu = 1$ different value of μ and absolute error at for Example 1 when $x = 0.5.$

τ	$\vartheta_6(x,\tau)$ $\mu = 0.75$	$\vartheta_6(x,\tau)$ $\mu = 0.9$	$\vartheta_6(x,\tau)$ $\mu = 1$	$\vartheta(x,\tau)$ *Exact*	*Abs. Erorr*
0.25	0.374305592	0.338496334	0.321006351	0.321006354	3.3×10^{-9}
0.5	0.502687021	0.443048288	0.412179904	0.412180317	4.13×10^{-7}
0.75	0.657729920	0.574822837	0.529242706	0.529250004	7.298×10^{-6}
1	0.846177885	0.742086432	0.679513888	0.679570457	5.6569×10^{-5}

Table 2. Numerical results of the 6^{th} -SVIM and exact solutions for different value of μ and absolute error at $\mu = 1$ for Example 2 when $x = y = \pi/4.$

τ	$\vartheta_6(x,y,\tau)$ $\mu = 0.75$	$\vartheta_6(x,y,\tau)$ $\mu = 0.9$	$\vartheta_6(x,y,\tau)$ $\mu = 1$	$\vartheta(x,y,\tau)$ *Exact*	*Abs. Erorr*
0.25	0.347153805	0.372697656	0.389400397	0.389400391	6×10^{-9}
0.5	0.276807506	0.291306665	0.303266059	0.303265329	7.30×10^{-7}
0.75	0.230207127	0.232303327	0.236195373	0.236183276	1.2097×10^{-5}
1	0.194841929	0.188031784	0.184027777	0.183939720	8.8057×10^{-5}

Table 3. Numerical results of the 6^{th} -SVIM and exact solutions for different value of μ and absolute error at $\mu = 1$ for Example 3 when $x = y = z = 0.5$.

τ	$\vartheta_6(x,y,z,\tau)$ $\mu = 0.75$	$\vartheta_6(x,y,z,\tau)$ $\mu = 0.9$	$\vartheta_6(x,y,z,\tau)$ $\mu = 1$	$\vartheta(x,y,z,\tau)$ *Exact*	*Abs. Erorr*
0.25	0.000121392	0.000086422	0.000069342	0.000069342	0
0.5	0.000246764	0.000188523	0.000158378	0.000158379	$1E\times10^{-9}$
0.75	0.000398173	0.000317209	0.000272697	0.000272705	8×10^{-9}
1	0.000582204	0.000480553	0.000419447	0.000419502	5.5×10^{-8}

5. Numerical and Graphical Discussions

In this part, we demonstrate the good agreement between exact and approximate solutions resulting from using SVIM. This concord is visible in the discussions of graphs and the analysis of tabulated numerical results for Examples 1, 2 and 3. Tables 1, 2, and 3 illustrate the exact and approximate solutions, with absolute error, for the 1-D, 2-D, and 3-D fractional heat-like equations in Examples 1, 2, and 3, respectively, at different values of μ. Also, figures 1, 2, 3, 4, 5, and 6 illustrate the exact and approximate solutions for the 1-D, 2-D, and 3-D fractional heat-like equations in Examples 1, 2, and 3, respectively, at different values of μ.

The results obtained in Examples 1, 2, and 3 are comparable to those obtained from the optimal homotopy analysis method, the natural transform method, and the Laplace variational iteration method. Finally, the results conclude that the SVIM is an excellent refinement of existing numerical techniques.

6. Conclusion

In this paper, the Sumudu Variational Iteration Method (SVIM) has been effectively and convergently applied to fractional heat-like equations involving the Atangana-Baleanu-Caputo operator. The approximate solutions for one-dimensional (1-D), two-dimensional (2-D), and three-dimensional (3-D) fractional heat-like equations with this operator converge to the exact solutions. The results obtained using SVIM are compared favorably with those derived from the Optimal Homotopy Asymptotic Method (OHAM), the Numerical Transformation Method (NTM), and the Linear Variational Iteration Method (LVIM). These comparisons demonstrate that SVIM is a significant improvement over existing numerical technique.

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