مجلة العلوم البحتة والتطبيقية
Journal of Pure & Applied Sciences
www.Suj.sebhau.edu.ly ISSN 2521-9200
Received 08/01/2018 Revised 08/02/2018 Published online 30/06/2018

Randomness Effect on a Fish Population Model Using Gumbel Distribution

*Almbrok Hussin Alsonosi Omar , Iman Aissa Alghannay Ahmed
Mathematics Department, Faculty of Sciences/University of Sebha, Libya
*Corresponding author: alm.omar@sebhau.edu.ly

Abstract Uncertainty in mathematical models has been raised in the last period. It can be happened due to insufficient knowledge about particular components of the model. Probability theory is a powerful tool can be applied to treat many of these problems. In this work, the effects of randomness on the fish population model are studied. The Gumbel distribution is chosen to explore the uncertainty in the model by considering the initial population sizes and the harvesting parameter of the deterministic model as the located parameter of the distribution. Discussion is given and the stability of the equilibrium points is studied. Furthermore, the random behaviour of the population and its statistical properties are investigated and compared with the deterministic behaviour.

Keywords: Fish population model, Gumbel distribution, Randomness, Uncertainty, Stability.

دراسة تأثير العشوائية على مجتمع الأسماك بأستخدام توزيع جامبل

*المبروك حسين السنوسي عمر و إيمان عيسى الغناي أحمد
قسم الرياضيات- كلية العلوم- جامعة سبها، ليبيا
*للمراسلة: alm.omar@sebhau.edu.ly

الملخص مفهوم عدم اليقين في النماذج الرياضية يحدث في أغلب الأحيان عندما تكون المعلومات حول المجتمع المستهدف من الدراسة غير واضحة بما فيه الكفاية. بمعنى ان قيم المعاملات التي تكون النموذج غير دقيقة. نظرية الاحتمالات تعتبر من الادوات الفعالة والقوية للتعامل مع المشاكل التي يسببها النقص في هذه المعلومات. في هذا الورقة، درسنا تأثير العشوائية على نموذج مجتمع سمكي بأستخدم توزيع جامبل الاحتمالي. وذلك بفرض ان مركز التوزيع يمثل قيمة حجم المجتمع الابتدائي وبنفس الطريقة تم فرض كمية الحصاد للسمك. تم دراسة السلوك العشوائي لمجتمع السمك تحت هذه الفرضيات وكذلك دراسة الخصائص الاحصائية لسلوك المجتمع حيث تمت مقارنة السلوك العشوائي للنموذج مع سلوك النوذج الواقعي.

الكلمات المفتاحية: نموذج مجتمع سمكي، توزيع قامبل، العشوائية، عدم اليقين، الإستقرار.

Introduction

Differential equations is a powerful tool for modelling natural phenomena and explaining the behavior of the changing in dynamic systems. It is used to find out how the population changes over time. In instance, the fish population in one of the Great Lakes can be mathematically molded by given birth, death, harvesting rates and the initial population size. The parameters and the initial population size are usually estimated as real values. In reality, mathematical modelling scenarios involve an inherent level of uncertainty. It can happen due to insufficient knowledge about components of the model and the proper values for the parameters as well as the assumptions adopted during the modelling process are not completely true. In system theory, uncertainty is classically treated in probabilistic form by the theory of stochastic processes. Indeed, there are many ways to incorporate randomness into quantitative and qualitative modelling. In the literature, the effect of randomness in mathematical models discussed by [1-4]. The authors adopt different random distributions and various approaches to introduce the random uncertainty in the models. The effect of considering the initial conditions of the models is also investigated by [5-7] where the beta distribution chosen to introduce the uncertainty in SI and SIR epidemic models. The influence of uncertain initial sizes of predator prey model by treating the uncertainty using the tools of probability theory is investigated in [8, 9] and by using the tools of the fuzzy set theory is studied in [10, 11].

In this paper, Gumbel distribution is considered as the initial fish population size and the harvesting amount according to the case study. The influence of the consideration is analytically explained in some cases when the probability density function of the random solution is obtained. Besides, numerical simulation is carried out in other cases.

Deterministic model

The change of the fish population that lives in a great lake can be described mathematically given birth, death and harvesting rates, by using differential equations as [12]

$$\frac{dy}{dt} = Birth\ rate - Death\ rate - Harvest\ rate \quad (1)$$

Denoting the fish population by $y(t)$, Birth rate by $by(t)$, death rate by $(m + cy(t))y(t)$ and harvest rate by H.

$$\frac{dy}{dt} = by(t) - (m + cy(t))y(t) - H \quad (2)$$

Or

$$\frac{dy}{dt} = ay - cy^2 - H, \quad y(t_0) = y_0 \quad (3)$$

Where $a = b - m$ and c are nonnegative. Fish immigration and emigration rates from rivers that meet the lake are supposed cancel each other out.

Three cases were assumed to study the population. First, no overcrowding, that means $c = 0$. The equation (3) becomes as follows

$$\frac{dy}{dt} = ay - H, \quad y(t_0) = y_0 \qquad (4)$$

The solution of equation (4) is

$$y(t) = \frac{H}{a} + \left(y_0 - \frac{H}{a}\right) e^{at} \qquad (5)$$

The equilibrium point H/a is the solution of equation (4) at $y_0 = H/a$. Otherwise the population grows without bound or becomes extinct where $y_0 > H/a$ and $y_0 < H/a$ respectively. The extinct time t^* is

$$t^* = \ln\left(\frac{H}{H - ay_0}\right)^{1/a} \qquad (6)$$

The second case, harvesting is not considered. Then equation (3) becomes

$$\frac{dy}{dt} = ay - cy^2, \quad y(t_0) = y_0 \qquad (7)$$

Which has the solution

$$y(t) = \frac{a}{c + \left(\frac{a - cy_0}{y_0}\right) e^{-at}} \qquad (8)$$

The points $y = 0$ and $y = a/c$ are equilibrium where the solution $y = a/c$ seems to attract all other non-constant solution curves.

The third case described by equation (3) where the overcrowding and harvesting are taken into account. The amount of the fish population over time is

$$y(t) = \frac{a + q\tan\left(\frac{1}{2}\left(2\tan^{-1}\left(\frac{aq - 2cqy_0}{-q^2}\right) - qx\right)\right)}{2c} \qquad (9)$$

Where $q = \sqrt{4cH - a^2}$. The equilibriums of equation (3) are

$$y = \frac{a}{2c} \pm \frac{c}{2}\sqrt{a^2 - 4cH} \qquad (10)$$

Where $a^2 > 4cH$.

Gumbel Distribution

The Gumbel distribution is perhaps the most widely applied statistical distribution for problems in engineering, it is the most common type of Fisher-Tippett extreme value distributions. It is useful in predicting the chance that an extreme earthquake, flood or other natural disaster will occur. The probability density function (PDF) is

$$p(x; \alpha, \beta) = \frac{Exp\left[\frac{x - \alpha}{\beta} - e^{\frac{x-\alpha}{\beta}}\right]}{\beta} \qquad (11)$$

The cumulative distribution function (CDF) is

$$c(x; \alpha, \beta) = Exp\left[-e^{-\left(\frac{x-\alpha}{\beta}\right)}\right] \qquad (12)$$

Where, α and β are the location and the scale parameters respectively. This distribution is selected in this study to be the initial fish population size and the harvesting parameter. The reason behind that is the symmetry of the distribution that helps us to study the behaviour of the population on the neighborhood of the crisp values. Moreover, it is also useful when we consider the parameters of the model may have minimum or maximum extreme values.

Random Model

In the deterministic fish population mode the initial state of the population and harvesting rate are assumed to be non-negative number. Here, the uncertainty is introduced in this model by considering the intimal condition and harvesting rate as random have particular distribution function. The central location of the distribution is set to be the crisp initial state and harvesting rate of the deterministic model. The reason is to study the behavior of the population beside these values. In this study, the Gumbel distribution is chosen to present the uncertainty in the model. By these assumptions equations (3), (4) and (7) become

$$\frac{dy}{dt} = ay - cy^2 - H, \quad y(t_0) = Y_0 \qquad (13)$$

$$\frac{dy}{dt} = ay - H, \quad y(t_0) = Y_0 \qquad (14)$$

$$\frac{dy}{dt} = ay - cy^2, \quad y(t_0) = Y_0 \qquad (15)$$

Where $Y_0 \sim Gumbel[\alpha, \beta]$ and $H \sim Gumbel[\alpha_1, \beta_1]$, where α, β, α_1 and β_1 are the parameters of the chosen distribution. According to the new assumptions and considering the three cases of the deterministic model, the properties of the distributions of the fish population over time and the influence of the randomness will be discussed in the following.

1. Harvesting and No Overcrowding

Given the random initial size Y_0 and a crisp value H, and using transformation on (5) which is linear, the distribution fish population $y(t)$ over time will also have the same type of the initial distribution with new location $(H + (a\alpha - H)\, e^{at})/a$ and new scale parameter $e^{at}\beta$. The probability density function of $y(t)$ is defined over time given by

$$p(y; t, \alpha, \beta, a, H) = \frac{1}{\beta}\left[Exp\left[Exp\left[\frac{H + (ay - H)e^{-at} - a\alpha}{a\beta}\right] + \frac{e^{-at}\left(ay - H + e^{at}\left(H - a(\alpha + a\beta t)\right)\right)}{a\beta}\right]\right] \qquad (16)$$

The effect of a location parameter is to shift a distribution in one direction or another, while the effect of the scale parameter on a distribution is reflected in the shapes of the PDF. In terms of studying the density (16), with a low H the behavior of the random solution will unlimited grow. Besides, the low population and high harvesting the located and scale parameters of random solution will end up to zero in short time. Fig. 1 illustrates that, the density curves corresponding to the higher value of β have smaller tails than the curves corresponding to the lower values of β. That means, the values is closer to the distribution center for the smaller values of the scale parameter. The expectation and the variance of fishes can be found using the integration on (16) which are the moments of Gumbel distribution. In this case, the behaviors of the expectation and the variance will exponentially increase over time if the fishing rate is slightly small or there is not fishing (fishing rate equal zero).

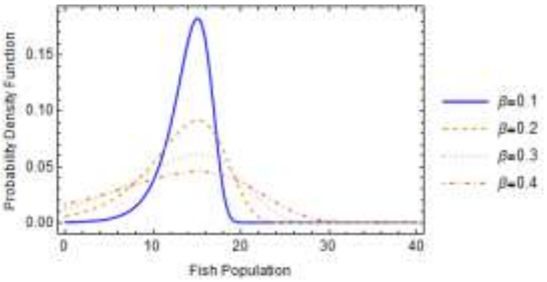

Fig.1: The PDF of the fish population at ($t = 2, a = 1, H = 5/3$).

Fig. 2 and Fig. 3 show the curves of the moments of the random solution for four different scale values. Each of these curves has the same location, where unlikely the expectation, the variance is small for the small values of β.

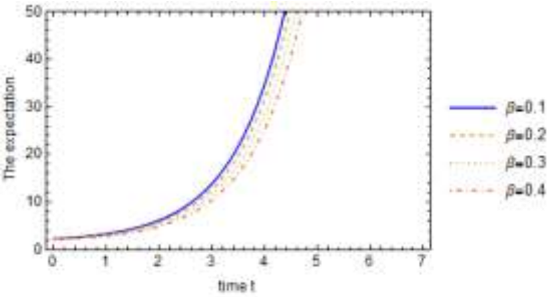

Fig. 2: The expectations of the random solution over time with four different scales ($a = 1, H = 5/3$).

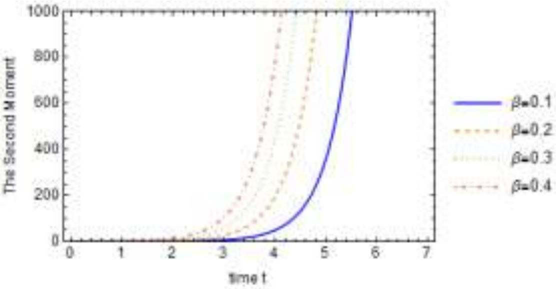

Fig. 3: The variances of the random solution over time with four different scales ($a = 1, H = 5/3$).

In general, the random behavior of the fish population will unboundedly grow if the initial distribution is above the equilibria H/a. Otherwise will soon end up with extinction. Equation (6) can calculate the extinction time which is real. As the effect of our considerations the extinction time will be a range of values described by density function as seen in Fig. 4.

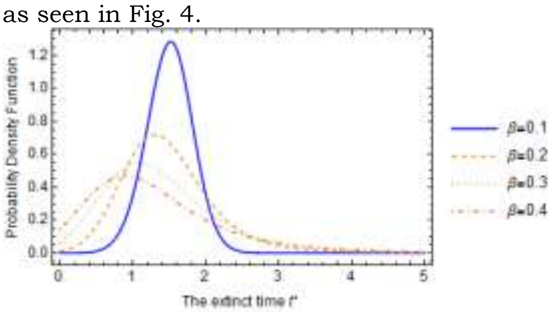

Fig. 4: The PDF of the extinction time for four different β. ($a = 1, H = 5/3$).

The behavior of the random solution of the fish population will have the same scenario if we set $H \sim Gumbel[\alpha_1, \beta_1]$ and $Y_0 \in \mathbb{R}_+^*$. The reason behind that is the random population model will not lost the linearity over time.

2. Overcrowding and No Harvesting

When the harvesting term is dropped and the overcrowding term is considered. The random model will take the form of equation (14). Unless the equation (8) is the solution of the problem (14), it is not particularly easy to derive the density probability function. So the numerical simulation is needed to describe the random solution. The approach that introduced in [13] is used to plot the approximate random solution curve and its statistical property. Fig. 5 shows the PDF of the population for fixed location and four different scale parameters. The verity of parameters is chosen to examine the effect of closeness and spacing of the values from the crisp value. We can see that for lower β the population is close to the located parameter and diverges when β increases. In another word, the uncertainty is low for the small value of β, and it is raised up for the high values of β.

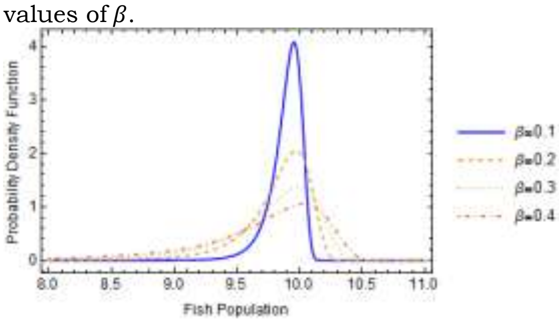

Fig. 5: The PDF of the fish population at ($t = 2,\ c = 1/12,\ a = 1$).

The expected behavior will decrease over time when the initial distribution is above the equilibria, and it will increase if the distribution started slightly under the equilibrium point. That means, the random solution will attracted to the non-zero equilibrium point a/c which is randomly stable. (See Fig. 6).

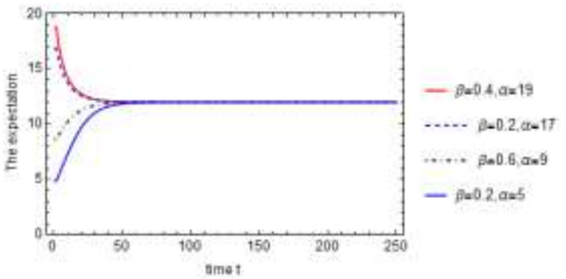

Fig. 6: The expectations of the random solution over time for four different locations and scales ($a = 1, c = 1/12$).

This fact can be concluded from the behavior of the variance of the random solution which will approach to zero as time goes up (see Fig 7).

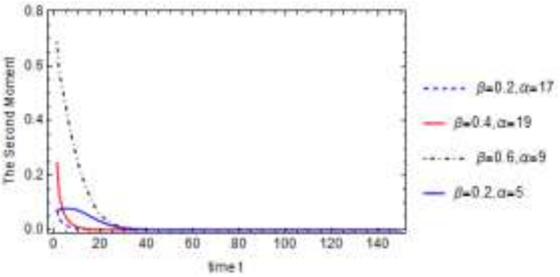

Fig. 7: The variances of the random solution for four different locations and scales ($a = 1, c = 1/12,$).

3. Overcrowding and Harvesting

Including the external factor (harvesting) and the internal control factor (overcrowding rate) in the random fish population is described by the equation (13). These consideration made the model more complicated and closer to the reality. The analytical solution of the deterministic fish model is displayed by equation (9). In terms of the initial fish population size is distributed as Gumbel distribution and $H \in \mathbb{R}_{+}^{*}$, the fish population will randomly affected similar to the second case. The random solution will attract to the upper equilibrium point that described in equation (10) provided that the located and the scale parameters ensure the distribution function of the initial fish population is greater than the lower equilibrium point. This scenario will happen with light harvesting. In general, the random population could be forced to extinction over time when harvesting level is high or in the case the distribution function of the population is lied under the lower equilibrium point even the harvesting is low. The more interested case when the initial fish population and harvesting rate are both random have particular Gumbel distribution functions. In this case numerical simulation is carried out to study the behavior of the fish population over time. Fig. 8 displayed the curves of the expected values of the population over time for different transactions of the initial Gumbel distribution and for $H \sim Gumbel(5/3\ , 0.1)$. The expectations are approached to the equilibrium point as time increases whether the located parameters are less than or greater than the lower equilibria.

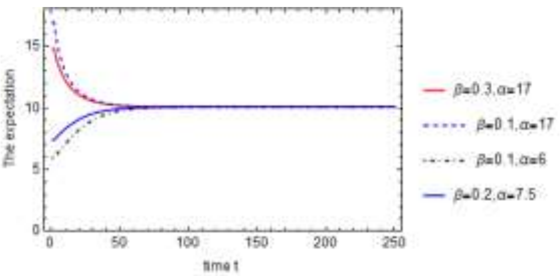

Fig. 8: The expectations of the random solution over time for four different locations and scales ($a = 1, c = 1/12$).

The second moments of the fish distributions over time is presented in Fig. 9. The figure illustrates that, for the high initial scale the curves will decrease over time and then increase to fixed value. Besides, the uncertainty is increased as time goes up for some small initial values of scales and then becomes fixed. Indeed, the distribution of the fish population is also affected by the uncertainty in the amount of the harvesting. Considering the uncertainty in the fish population model allows us to predict the earliest time at which the population will reach the given equilibria. In addition, the distribution of the population and the harvesting is allowed us to determine the amount of the fish population not as a real value but as a range of values described by PDFs.

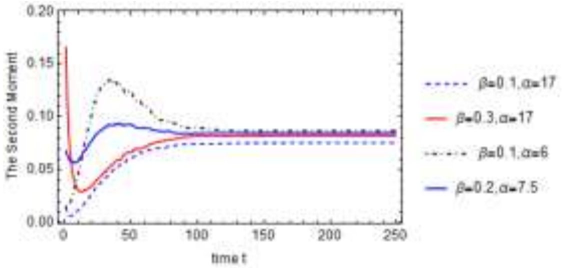

Fig. 9: The variances of the random solution for four different locations and scales ($a = 1, c = 1/12$).

Studying the behaviour of the fish population as random model helps us to make clearer conclusion on the situation of population size over time, than studying the deterministic model. That is, whenever the initial size or the number of harvesting rate is, the changing of these values over time are bounded and can easily described using the PDFs and their statistical proprieties.

Conclusion

In the deterministic model, the initial condition and the factors the used to build the model are often assumed to be crisp values which is uncertain and not accrued. In this paper, the uncertainty was treated in the fish population model by assuming the Gumbel distribution as the initial population size and the harvesting rate. The influence of the randomness on the behaviour of the population was studied over time. The ways in which the parameters of this initial Gumbel distribution affect the expectation and the variance over time is examined in the three cases. It found that, the stable equilibrium points attracted the random solution whatever the scale parameters provided that the range of the initial distribution must lie in specific domain. That means, the deterministic stable point would be random stable. In those cases, the expectation approached the equilibrium points the when variance converged to zero. In addition, introducing the uncertainty in the fish model affected the extinction time which would be uncertain described by probability density function depends on the chosen of the parameters of the initial size. In conclusion, the effect randomness is made us deal with a range of value instead of crisp value.

References

[1]- G. Gonzaléz, L. Jódar, R. Villanueva, and F. Santonja, "Random modeling of population dynamics with uncertainty," WSEAS TRANSACTIONS on BIOLOGY and MEDICINE, vol. 5, pp. 34-45, 2008.

[2]- P. K. Pollett, A. H. Dooley, and J. V. Ross, "Modelling population processes with random

initial conditions," Mathematical Biosciences, vol. 223, pp. 142-150, 2010.
[3]- T. Soong, Random differential equations in science and engineering: Academic Press, 1973.
[4]- P. N. V. Tu and E. A. Wilman, "A generalized predator- prey model: Uncertainty and management," Journal of Environmental Economics and Management, vol. 23, pp. 123-138, 1992.
[5]- B. Kegan and R. W. West, "Modeling the simple epidemic with deterministic differential equations and random initial conditions," Mathematical Biosciences, vol. 195, pp. 179-193, 2005.
[6]- A. H. A. Omar and Y. Abu-Hasan, "Numerical Simulations of the SIR Epidemic Model with Random Initial States," International Conference on Computer Engineering & Mathematical Sciences, 2012.
[7]- A. H. A. Omar and Y. A. Hasan, "Numerical simulations of an SIR epidemic model with random initial states," SCIENCEASIA, vol. 39, pp. 42-47, 2013.
[8]- A. H. A. Omar and Y. Abu-Hasan, "Uncertainty in the Initial Population of a Predator-Prey Model," In Proceedings of 2nd International Conference on Mathematical Sciences, pp. 606-613, 2010.
[9]- A. H. A. Omar and Y. Abu-Hasan, "The Interaction of Predator Prey with Uncertain Initial Population Sizes " Journal of Quality Measurement and Analysis (JQMA), vol. 7, pp. 75-83, 2011.
[10]- A. H. A. Omar and I. A. A. Ahmed, "Stability of Fuzzy Lotka Volterra Model using Lyapunov Function," Journal of Academic Research, pp. 601-616, 2016.
[11]- A. H. A. Omar, I. A. A. Ahmed, and Y. A. Hasan, "The Fuzzy Ratio Prey-Predator Model," International Journal of Computer Science and Electronics Engineering (IJCSEE) vol. 1, p. 3, 2015.
[12]- R. L. Borrelli and S. C. Coleman. "Differential equations: A modeling perspective". Vol. 2. New York: Wiley, 2004.
[13]- A. H. A. Omar and I. A. A. Ahmed, "Fifth Order Improved Runge-Kutta Method for Random Initial Value Problems," Journal of Pure & Applied Sciences (JOPAS), vol. 16, 2016.