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Numerical Solution of Differential Equations Using the Wavelet-Based Galerkin Method with Fibonacci Wavelets

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A B S T R A C T

Differential equations form the foundation of scientific theories that address numerous real-world physical challenges. Numerical methods enable the resolution of complex problems through relatively simple operations. A significant advantage of numerical methods, compared to analytical methods, is their ease of implementation on modern computers, allowing for quicker solutions. Galerkin's method belongs to a broader category of numerical techniques. Additionally, wavelet analysis represents a promising domain within applied and computational research. This paper establishes a wavelet-based Galerkin method for numerically solving differential equations, utilizing Fibonacci wavelets as trial functions. The proposed method yields results comparable to existing techniques and provides solutions that closely approximate exact answers for certain problems, thereby demonstrating its effectiveness and accuracy.

الحل العددي للمعادلات التفاضلية باستخدام طريقة غاليركين المعتمدة على المويجات مع مويجات فيبوناتشي

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الكلمات المفتاحية:
المعادلات التفاضلية
طريقة غاليركين
مويجات فيبوناتشي
دالة البقايا الموزونة

الملخص

تشكل المعادلات التفاضلية الأساس للنظريات العلمية التي تعالج العديد من التحديات الفيزيائية في العالم الحقيقي. تتيح الطرق العددية حل المشكلات المعقدة من خلال عمليات بسيطة نسبيًا. تتمثل ميزة رئيسية للطرق العددية، مقارنة بالطرق التحليلية، في سهولة تنفيذها على الحواسيب الحديثة، مما يتيح إيجاد الحلول بشكل أسرع. تنتمي طريقة غاليركين إلى فئة أوسع من التقنيات العددية. بالإضافة إلى ذلك، تمثل تحليل المويجات مجالًا واعدًا في البحوث التطبيقية والحاسوبية. يقدم هذا البحث طريقة غاليركين تعتمد على المويجات لحل المعادلات التفاضلية عددياً، باستخدام مويجات فيبوناتشي كدوال تجريبية. تُظهر الطريقة المقترحة نتائج مماثلة للتقنيات الحالية وتوفر حلولًا تقارب الحلول الدقيقة لبعض المشكلات، مما يؤكد فعاليتها ودقتها.

1. Introduction:

Numerous challenges involving various linear and nonlinear problems exist within the fields of science and engineering. Specifically, second-order differential equations, which are subject to a variety of boundary conditions, can be addressed through either analytical or numerical methods. In the fields of engineering science and applied mathematics, numerical simulation has emerged as a vital instrument for modeling physical phenomena, especially in instances where analytical solutions are either unavailable or exceedingly difficult to derive.
The literature on differential equation resolution reveals that many researchers have sought to achieve higher accuracy in a timely manner by employing numerical methods. It is important to note that analytical solutions to such boundary value problems are rarely attainable. A range of methods for the numerical solution of differential equations is documented in existing literature [1–4]. Additionally, some numerical methods for solving different types of such problems using Fibonacci wavelets are available in the literature [5–7].
Wavelets have emerged as independent concepts across various disciplines, including mathematics, quantum physics, electrical engineering, and seismic geology. A key principle in approximation theory is the representation of a smooth function as a series expansion utilizing orthogonal polynomials. Currently, the exploration of wavelet function bases is regarded as a promising alternative to

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traditional piecewise polynomial trial functions in the finite element analysis of differential equations. The Galerkin method is highly esteemed in applied mathematics for its efficiency and practicality [8–9]. The Galerkin method utilizing wavelets offers significant advantages over traditional finite difference and finite element methods, resulting in extensive applications across various fields of science and engineering. To some degree, the wavelet approach serves as a formidable alternative to the finite element method. Furthermore, the wavelet technique presents an effective alternative for the numerical solution of differential equations.

This research introduces the wavelet-based Galerkin method for numerically solving differential equations utilizing Fibonacci wavelets (FWGM). The methodology entails expressing the solution in terms of Fibonacci wavelets, which are defined by unknown coefficients. By exploiting the characteristics of Fibonacci wavelets alongside the Galerkin method, we can ascertain these unknown coefficients, thereby achieving the numerical solution of the differential equations.

The organization of the paper is delineated as follows: Section 2 discusses Fibonacci wavelets and their application in function approximation. Section 3 examines the wavelet-based Galerkin method using Fibonacci wavelets. Section 4 offers a numerical example. Finally, Section 5 provides a discussion of the conclusions drawn from the research undertaken.

2. Fibonacci wavelets and Function Approximation:

2.1. Fibonacci Polynomials: The general definition of Fibonacci polynomials [10 - 11] is as follows:

$$\tilde{F}_m(x) = \begin{cases} 1, & m = 0 \\ x, & m = 1 \\ x\tilde{F}_{m-1}(x) + \tilde{F}_{m-2}(x), & m > 1 \end{cases} \quad (2.1)$$

Additionally, these polynomials can be expressed in the form of powers as shown:

$$\tilde{F}_m(x) = \sum_{i=0}^{\frac{m}{2}} \binom{m-i}{i} x^{m-2i}, \quad m > 0 \quad (2.2)$$

Also, if $\tilde{F}_m(x)$, $m = 0, 1, \ldots, M-1$ are Fibonacci polynomials, then

$$\int_0^1 \tilde{F}_m(x)\tilde{F}_n(x)\,dx = \sum_{i=0}^{\frac{m}{2}} \sum_{j=0}^{\frac{m}{2}} \binom{m-i}{i}\binom{m-i}{j}\left(\frac{1}{m+n-2i-2j+1}\right) \quad (2.3)$$

2.2. Fibonacci Wavelets: Fibonacci wavelets [10-11] are defined in the following manner:

$$\psi_{n,m}(x) = \begin{cases} 2^{\frac{k-1}{2}} \dfrac{\hat{F}_m(2^{k-1}x - \hat{n})}{\sqrt{W_m}}, & \dfrac{\hat{n}}{2^{k-1}} \le x < \dfrac{\hat{n}+1}{2^{k-1}}, \\ 0, & \text{otherwise}, \end{cases} \quad (2.4)$$

In which $\hat{F}_m(x) = \dfrac{1}{\sqrt{W_m}} \tilde{F}_m(x)$ with

$$W_m(x) = \int_0^1 \left\{\tilde{F}_m(x)\right\}^2 dx$$

where W_m , for $m = 0,1,2,\ldots,M-1$ are obtained by Eq. (2.3), and m denotes the order of the Fibonacci polynomials and , $n = 1,2,\ldots,2^{k-1}, k \in N$.

For instance, for $k = 1$ and $M = 3$, the Fibonacci wavelet bases as given below:

$$\psi_{1,0}(x) = 1 ,$$

$$\psi_{1,1}(x) = \sqrt{3}\,x ,$$

$$\psi_{1,2}(x) = \frac{1}{2}\sqrt{\frac{15}{7}}\left(1 + x^2\right)$$

and so on.

2.3. Function Approximation:

Let's assume $y(x) \in L^2[0, 1)$ can be expressed using Fibonacci wavelets in the following manner:

$$y(x) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c_{n,m}\psi_{n,m}(x) \quad (2.5)$$

By cutting off the infinite series mentioned above, we

$$y(x) = \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} c_{n,m}\psi_{n,m}(x) \quad (2.6)$$

3. Method of Solution:

Consider the differential equation of the form,

$$y'' + \alpha y' + \beta y = f(x) \quad (3.1)$$

With boundary conditions $y(0) = a, \; y(1) = b$ (3.2)

Here, α & β are constants, while $f(x)$ is a continuous function.

Write the Eq. (3.1) as

$$R(x) = y'' + \alpha y' + \beta y - f(x) \quad (3.3)$$

where $R(x)$ is the residual of the Eq. (3.1) and it is zero the exact solution is known and the boundary conditions are met.

The trial series solution of Eq. (3.1), $y(x)$ defined as $[0, 1)$ satisfies the specified boundary conditions and can be extended to a modified Fibonacci wavelet with unknown parameters as follows:

$$y(x) = \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} c_{n,m}\psi_{n,m}(x) \quad (3.4)$$

where $c_{n,m}$'s are unknown coefficients and are to be determined.

The accuracy of the solution is improved by opting for higher degree Fibonacci wavelet polynomials. To obtain the values of the second derivative of Eq. (3.4), one must differentiate it twice w.r.t. x and substitute these values $y(x)$, $y'(x)$, $y''(x)$ in Eq. (3.3). The unknown parameters $c_{n,m}$'s can be determined by choosing weight functions as the assumed basis elements and conducting integration on the boundary values, along with the residual, to ensure that it equals zero [12].

i.e. $\int_0^1 \psi_{1,m}(x)\, R(x)\, dx = 0$, $m = 0, 1, 2, \ldots\ldots$

This enables us to formulate a system of linear algebraic equations. By solving this system, we can identify the unknown parameters. Following this, we can substitute these parameters into the trial solution, referred to as Eq. (3.4), which allows us to derive the numerical solution for Eq. (3.1).

In order to assess the precision of the FWGM concerning the test problems, we make use of error known as the maximum absolute error. The formula for calculating the maximum absolute error is as follows:

$$E_{\max} = \max\left| y(x)_e - y(x)_n \right|,$$

where $y(x)_e$ and $y(x)_n$ are exact and numerical solutions respectively.

4. Numerical Illustration:

Problem 4.1 First, consider the differential equation [10],

$$y'' + y = -x, \quad 0 \le x \le 1 \quad (4.1)$$

With boundary conditions: $y(0) = 0, \; y(1) = 0$ (4.2)

Eq. (4.1) should be implemented according to the method described in section 3:

Using Eq. (4.1), the residual is given as:

$$R(x) = y'' + y + x \tag{4.3}$$

Then, the weight function $w(x) = x(1-x)$ should be selected for Fibonacci wavelet bases in order to meet the specified boundary conditions Eq. (4.2),

i.e. $\psi(x) = w(x) \times \psi(x)$

$$\psi_{1,0}(x) = \psi_{1,0}(x) \times x(1-x) = x(1-x)$$

$$\psi_{1,1}(x) = \psi_{1,1}(x) \times x(1-x) = \left(\sqrt{3}\,x\right) x(1-x)$$

$$\psi_{1,2}(x) = \psi_{1,2}(x) \times x(1-x) = \frac{1}{2}\sqrt{\frac{15}{7}}\left(1 + x^2\right) x(1-x)$$

The trail solution of Eq. (4.1) for $k = 1$ and $m = 2$ is given by

$$y(x) = c_{1,0}\,\psi_{1,0}(x) + c_{1,1}\,\psi_{1,1}(x) + c_{1,2}\,\psi_{1,2}(x) \tag{4.4}$$

Now, Eq. (4.4) becomes

$$y(x) = c_{1,0}\left\{x(1-x)\right\} + c_{1,1}\left\{\left(\sqrt{3}\,x\right)x(1-x)\right\} + c_{1,2}\left\{\frac{1}{2}\sqrt{\frac{15}{7}}\left(1 + x^2\right)x(1-x)\right\} \tag{4.5}$$

Differentiating Eq. (4.5) w.r.t. x twice and put the values of y', y'' in Eq. (4.3) then we obtain the residual of Eq. (4.1). If the weight functions equivalent the basis functions in the trail solution, we can then proceed to consider the following using the weighted residual method:

$$\int_0^1 \psi_{1,j}(x)\, R(x)\,dx = 0, \quad j = 0, 1, 2 \tag{4.6}$$

For $j = 0, 1, 2$ in Eq. (4.6),

$$\text{i.e.} \quad \left.\begin{aligned} \int_0^1 \psi_{1,0}(x)\, R(x)\,dx &= 0 \\ \int_0^1 \psi_{1,1}(x)\, R(x)\,dx &= 0 \\ \int_0^1 \psi_{1,2}(x)\, R(x)\,dx &= 0 \end{aligned}\right\} \tag{4.7}$$

Based on Eq. (4.7), a set of algebraic equations involving unknown coefficients such as $c_{1,0}$, $c_{1,1}$ and $c_{1,2}$. Solving this system, obtained the values for $c_{1,0} = 0.2062$, $c_{1,1} = 0.1089$ and $c_{1,2} = -0.0240$. Once these values are found, they can be substituted into Eq. (4.5) to get the numerical solution. A comparison of the numerical solution and absolute errors is presented in Table 1, and the numerical solution and the exact solution of Eq. (4.1) $y(x) = \dfrac{\sin(x)}{\sin(1)} - x$ in Figure 1.

Table 1: Comparison of numerical solution and absolute error with the exact solution of problem 4.1

x	Numerical solution		Exact solution	Absolute error	
	Ref [13]	**FWGM**		**Ref [13]**	**FWGM**
0.1	0.0186708	0.0186588	0.0186420	2.88e-5	1.70e-5
0.2	0.0361655	0.0361048	0.0360977	6.78e-5	7.10e-6
0.3	0.0512714	0.0511642	0.0511948	7.66e-5	3.10e-5
0.4	0.0628316	0.0627351	0.0627829	4.87e-5	4.80e-5
0.5	0.0697452	0.0697491	0.0697470	1.84e-6	2.10e-6
0.6	0.0709672	0.0709857	0.0710184	5.12e-5	3.30e-5
0.7	0.0655087	0.0655327	0.0655851	7.64e-5	5.20e-5
0.8	0.0524367	0.0525260	0.0525025	6.58e-5	2.40e-5
0.9	0.0308742	0.0309247	0.0309019	2.77e-5	2.30e-5

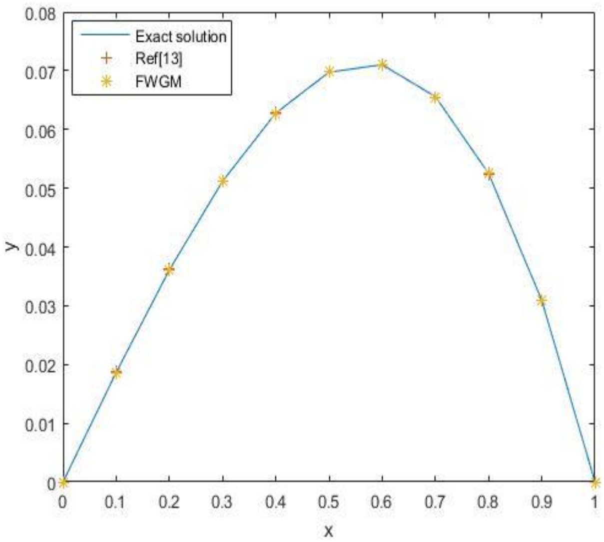

Fig. 1: Comparison of the numerical solution with the exact solution for problem 4.1.

Problem 4.2 Next, another differential equation [13],

$$y'' - \pi^2 y = -2\pi^2 \sin(\pi x), \quad 0 \le x \le 1 \tag{4.8}$$

With boundary conditions: $y(0) = 0, \quad y(1) = 0$ (4.9)

In accordance with section 3 and the previous problem, we find the values of $c_{1,0} = 6.7181$ $c_{1,1} = 2.0894$ and $c_{1,2} = -4.9480$. Substituting these values into Eq. (4.5), we obtain the numerical solution. Table 2 compares the numerical solution to the absolute errors and the figure 2 represents the numerical solution to the exact solution of Eq. (4.8) as $y(x) = \sin(\pi x)$.

Table 2: Comparison of numerical solution and absolute error with the exact solution of problem 4.2

x	Numerical solution		Exact solution	Absolute error	
	Ref [13]	**FWGM**		**Ref [13]**	**FWGM**
0.1	0.310207	0.3079992	0.309016	1.19e-3	1.02e-03
0.2	0.589551	0.5880739	0.588772	7.79e-4	7.00e-04
0.3	0.809478	0.8094184	0.809016	4.62e-4	4.00e-04
0.4	0.949592	0.9515192	0.951056	1.46e-3	4.60e-04
0.5	0.997656	1.0001543	1.000000	2.34e-3	1.50e-04
0.6	0.949592	0.9513935	0.951056	1.46e-3	3.40e-04
0.7	0.809478	0.8092985	0.809016	4.62e-4	2.80e-04
0.8	0.589551	0.5878225	0.587785	1.77e-3	3.80e-05
0.9	0.310207	0.3084107	0.309016	1.02e-3	6.10e-04

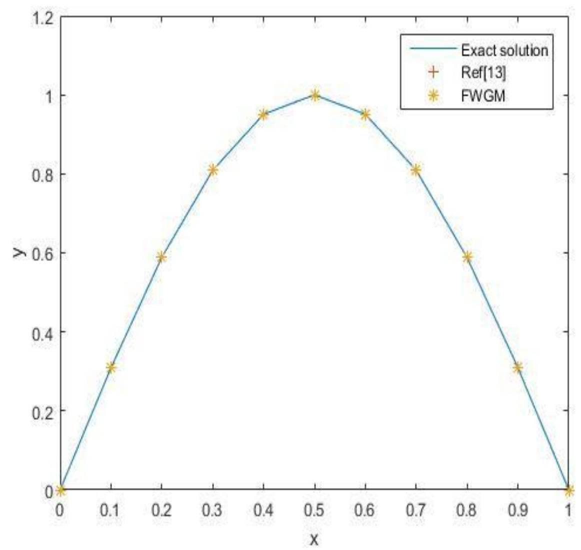

Fig. 2: Comparison between the numerical solution and the exact solution for problem 4.2.

Problem 4.3 Further, the differential equation [13]

$$y'' - y' = -\left(e^{x-1} + 1\right),\ 0 \le x \le 1 \qquad (4.10)$$

With boundary conditions: $y(0) = 0,\ \ y(1) = 0$ (4.11)

In accordance with section 3 and the previous problem, we find the values of $c_{1,0} = 0.5265$, $c_{1,1} = 0.1495$ and $c_{1,2} = 0.1437$. Substituting these values into Eq. (4.5), we obtain the numerical solution. Table 3 compares the numerical solution to the absolute errors and the figure 3 represents the numerical solution to the exact solution of Eq. (4.10) as $y(x) = x\left(1 - e^{x-1}\right)$.

Table 3: Comparison of numerical solution and absolute error with the exact solution of problem 4.3.

x	Numerical solution		Exact solution	Absolute error	
	Ref[13]	FWGM		Ref[13]	FWGM
0.1	0.059251	0.059376	0.059343	9.20e-5	3.30e-05
0.2	0.109902	0.110058	0.110134	3.32e-4	7.60e-05
0.3	0.150735	0.150954	0.151024	2.89e-4	7.00e-05
0.4	0.180249	0.180500	0.180475	2.26e-4	2.50e-05
0.5	0.196660	0.196861	0.196735	7.50e-5	1.26e-04
0.6	0.197904	0.197978	0.197808	9.60e-5	1.70e-04
0.7	0.181631	0.181540	0.181427	2.04e-4	1.13e-04
0.8	0.145212	0.144983	0.145015	7.00e-4	3.20e-05
0.9	0.085733	0.059376	0.085646	4.18e-5	3.30e-05

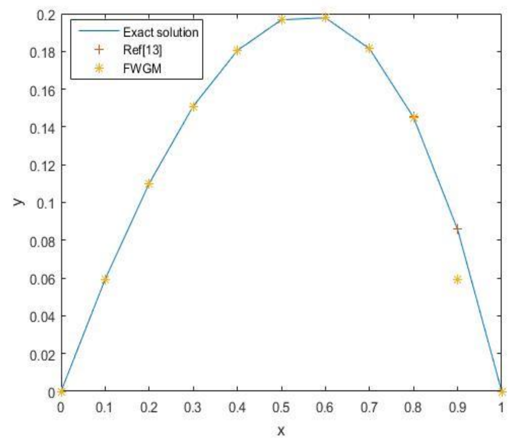

Fig. 3: Comparison between the numerical solution and the exact solution for problem 4.3.

Problem 4.4 Finally, the differential equation [14]

$$y'' - y^2 = 2\pi^2\cos(2\pi x) - \sin^4(2\pi x),\quad 0 \le x \le 1 \qquad (4.12)$$

With boundary conditions: $y(0) = 0,\ \ y(1) = 0$ (4.13)

In accordance with section 3 and the previous problem, we obtain the numerical solution and are presented with the exact solution of Eq. (4.12) as $y(x) = \sin^2(\pi x)$ in table 4 and figure 4.

Table 4: Comparison of numerical solution and absolute error with the exact solution of problem 4.4.

x	FWGM	Exact solution	Absolute error
0.1	0.096728	0.095492	1.24E-03
0.2	0.359769	0.345492	1.43E-02
0.3	0.659432	0.6545082	4.92E-03
0.4	0.909876	0.9045082	5.37E-03
0.5	0.998784	1	1.22E-03
0.6	0.910518	0.9045082	6.01E-03
0.7	0.657385	0.6545082	2.88E-03
0.8	0.348918	0.345492	3.43E-03
0.9	0.099824	0.095492	4.33E-03

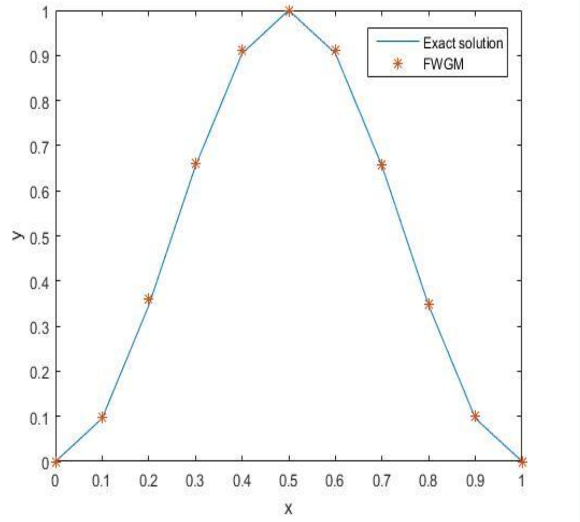

Fig. 4: Comparison between the numerical solution and the exact solution for problem 4.4.

5. Main Results:

The numerical solutions obtained from the proposed method, as demonstrated by the data, tables, and figures, indicate that this method yields results surpassing those of the existing method (Ref [13]) and shows a closer alignment with the exact solution. Additionally, the absolute error associated with this approach is significantly lower compared to the existing method (Ref [13]).

6. Conclusions:

This paper presents a wavelet-based Galerkin method for obtaining the numerical solutions of differential equations using Fibonacci wavelets (FWGM). This development significantly advances recent research in numerical analysis, providing substantial benefits to beginner researchers. The proposed method has been applied to several examples, producing commendable results in comparison to other well-established numerical techniques.

In conclusion, the proposed method has demonstrated exceptional effectiveness in the numerical solution of differential equations.

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