SEBHA UNIVERSITY JOURNAL OF PURE & APPLIED SCIENCES VOL.23 NO. 2 2024
DOI: 10.51984/JOPAS.V23I2.3590

مجلة جامعة سبها للعلوم البحتة والتطبيقية
Sebha University Journal of Pure & Applied Sciences

Journal homepage: www.sebhau.edu.ly/journal/jopas

The Applications of Fourier Series Harmonics in Musical Tones

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Keywords:
Fourier Series.
Frequency Domain.
Musical Tones.
Octaves.
Spectrum.
Time Domain.

A B S T R A C T

The Fourier Series is considered one of the most important computational tools in mathematics and has widespread usage, specifically in music. The present paper aims at presenting Fourier Series in the context with sound analysis and synthesis. Since Fourier Series decomposes complicated waves into simple sinusoids, that improves our approach to harmonics and thereby the synthesis of sounds. The present discussion how this mathematical method offers to musicians and sound engineers new approaches as to how to generate and evaluate musical tones and sounds. Analyzing various examples, this paper will help to explain the relationship between mathematics and music, with a focus on the role of Fourier analysis in modern music production and its role in creating and designing the new exceptional sound.

تطبيقات تو افقيات متسلسلة فوريرعلى النغمات الموسيقية

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الكلمات المفتاحية:
سلسلة فورير.
مجال التردد.
نغمات موسيقية.
طبقات صوتية.
الطيف.
المجال الزمني.

الملخص

تعتبر متسلسلة فورير احدى الطرقالحسابية المهمة في الرياضيات والتي لها استخدام واسع النطاق خاصة في الموسيقى. الورقة الحالية تهدف إلى تقديم متسلسلة فورير في سياق التحليل السليم والتوليف الصوتي. بما أن متسلسلة فورير تقوم بتحليل الموجات المعقدة إلى موجات جيبية بسيطة، فإن ذلك يحسن من أسلوبنا إتجاه التوافقيات وبالتالي تحليل الأصوات. المناقشة الحالية توضح كيف أن هذه الطريقة الرياضية تقدم للموسيقيين وكذلك مهندسي الصوت أساليب جديدة عن كيفية توليد وتقييم النغمات الموسيقية. من خلال تحليل الأمثلة المختلفة، هذه الورقة ستساعد في توضيح العلاقة بين الرياضيات والموسيقى مع التركيز على دور تحليل فورير في إنتاج الموسيقى الحديثة ودوره في خلق صوت استثنائي جيد.

1. Introduction

The Fourier Series, named after Joseph Fourier, is a critical mathematical tool in the analysis of periodic functions. It allows for the representation of a waveform as a sum of sine and cosine waves, providing a method to understand various scientific phenomena, including signal processing, quantum mechanics, acoustics, and many other fields. While Fourier Series has broad applications, its significance in music analysis is often underestimated. In music, it enables sound analysis and synthesis, allowing for the exploration of the structures that compose a musical piece. Musical tones can be described through Fourier Series as sums of sinusoidal functions, effectively describing the underlying principles of timbre, harmony, and the nature of fundamental frequencies combined with overtones.

This literature review explores how Fourier Series can aid in understanding and producing musical tones, particularly in relation to musical instruments and sound synthesis. Additionally, it examines how Fourier analysis contributes not only to the technical aspects of sound production but also to the creative processes of composing and performing music. The present paper begins by detailing Fourier Series and proceeds to illustrate its applications in the analysis of musical pieces, with the aim of enhancing the reader's appreciation for the relationship between mathematics and music. This analysis also demonstrates the potential of Fourier Series for modern sound design and music composition.

The concept of Fourier Series, envisioned by Jean-Baptiste Joseph Fourier in the early 1800s, is based on the fact that any waveform can be decomposed into sine and cosine components. This mathematical representation allows the breakdown of complex waveforms into their fundamental elements, which is crucial in disciplines such as acoustics

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Article History: Received 09 October 2024 - Received in revised form 10 November 2024 - Accepted 13 November 2024

and music. According to Bracewell [1], Fourier Series provide a rigorous framework for managing periodic signals, making them invaluable in sound analysis. They allow sound waves to be "dissected" into different frequencies, which is essential for the analysis of musical tones [2][3].

The use of Fourier Series in sound analysis is well-documented, particularly in isolating fundamental and overtone frequencies in musical tones. This separation aids researchers in understanding how different instruments produce sound. For instance, the work of K. S. M. B., K. D. A., S. M. A., and S. P. M. demonstrates how Fourier Series can be employed in harmonic analysis to understand how the fundamental frequencies of musical instruments interact with overtones. This principle can be expressed mathematically as follows:

$$f(t) = f_0 + \sum_{k=1}^{N} A_k \cos(k\omega_0 t + \phi_k)$$

where:

- (f_0) is the fundamental frequency,
- (A_k) are the amplitudes of the harmonics,
- (k) is the harmonic number,
- $(\omega_0 = \frac{2\pi}{T})$ is the angular frequency,
- (ϕ_k) represents the phase shifts of each harmonic.

Each of the practical aspects has been a significant advancement towards creating sounds with the help of Fourier Series in electronic music. It can perhaps be considered one of the earliest electronic instruments, as it uses Fourier analysis to produce various sorts of sounds. As noted by Chowning [4], many methods of sound synthesis, including additive synthesis, work from the premise of the Fourier Series, where musicians can combine sinusoidal waves to produce a complex timbre from sinusoidal components.

Apart from its application in sound synthesis, Fourier Series also impacts musical theory and composition. One significant point is that mathematical concepts can be employed to identify specific frequencies as components of musical notes; this way, a composer can use mathematical concepts to design new music. Lerdahl and Jackendoff [5] highlighted the cognitive elements of music processing, noting that Fourier analysis can inspire new compositional techniques and creative exploration.

Many studies have applied Fourier analysis to analyze individual instruments. For example, investigations conducted by Rossing et al. [6] on the sounds produced by stringed musical instruments, such as violins and cellos, illustrate the application of Fourier Series in analyzing overtone series. To support this premise, they presented their work as follows: Fourier analysis of overtones contributes to the characterization of sound timbre in string instruments.

The conventional Fourier Series has also found application in DSP (Digital Signal Processing) today, serving as a cornerstone for algorithms used in sound modulation. Techniques such as FFT have greatly advanced real-time sound processing, benefiting both solo musicians and ensemble performances, particularly among sound engineers. These advancements have broadened possibilities in sound design and music production [7].

2. Definition and Mathematical Formulation

When dealing with sequence convergence, the appearance of the sequence can deceive the eye. The terms of the sequence may seem to converge to a limit, yet they may not actually do so. For this reason, it is not sufficient to say that a sequence converges just because the terms seem to settle down to a value. We must have a criterion that not only guarantees convergence but also pronounces the sequence as converging when this happens [8] [9].

The Fourier Series can be mathematically expressed as:

Now, from the given Fourier series of

$$f(t) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{2\pi nt}{T}\right) + b_n \sin\left(\frac{2\pi nt}{T}\right) \right)$$

- (a_0) is the average value of the function,
- (a_n) and (b_n) are the coefficients for the cosine and sine components, respectively. These coefficients are determined using the following integrals:

- Average Value:

$$a_0 = \frac{1}{T}\int_0^T f(t)\, dt$$

- Cosine Coefficients:

$$a_n = \frac{2}{T}\int_0^T f(t) \cos\left(\frac{2\pi nt}{T}\right) dt$$

- Sine Coefficients:

$$b_n = \frac{2}{T}\int_0^T f(t) \sin\left(\frac{2\pi nt}{T}\right) dt$$

In the following sections, we will discuss pointwise convergence, uniform convergence, and the mean square convergence of sequences. We will give the reasons why it is important to understand not only what convergence looks like but also what it really is, particularly when we are dealing with sequence limits in the context of Fourier series. Pointwise convergence implies that as each value, within the domain is considered individually; the sequence of functions approaches a value (the function (g) in this instance). Put differently for any given (x) within the domains of the functions (f_n). (g) holds true;

$$\lim_{n\to\infty} f_n\,(x) = g(x)$$

This is also sometimes called "convergence for each point" or "convergence at each point." The limit function $(\ g\ )$ is then called the pointwise limit of the (f_n). On the other hand, if a sequence of functions converges uniformly to a limit function, it means that, as $(\ n\ )$ gets large, all the values $(f_n(x))$, for every $(\ x\ )$ in the domain, get to be nearly equal to $(g(x))$. [10][11]

$\forall\epsilon > 0\ \exists N$ such that $\forall n \geq N,\ \forall x \in \mathrm{dom}(f_n),\ |f_n(x) - g(x)| < \epsilon$

A uniform limit of continuous functions is always a continuous function. One important outcome, in Fourier analysis is known as Parseval's identity which demonstrates that the total of the squares of a functions Fourier coefficients is equivalent, to the integral of the function squared across its period length. Moreover, for a function expressed by means of a Fourier series Parseval's identity assert;

$$\sum (n = -\infty)^{(\infty)} |a_n|^2 = 1/(2\pi) \int (-\pi)^{(\pi)} |f(x)|^2 dx.$$

Parseval's identity is derived from the orthogonality of the basic Fourier series functions and is used to verify that the Fourier series converges to the function it represents.[12][13]

A function is said to be periodic if it repeats its values over time. A repeating function has a basic building block called a period, the amount of time (or the interval of space) it takes for the function to start over again. This basic repeating unit can be combined with other identical units to build a function that extends over an infinite amount of time (or space). One can also say that such a function has a basic rhythm. The period and the amount of time (or space) it takes for the function to run its course don't change; the basic unit structure of the function retains its form indefinitely. The Fourier series can be made to converge in several ways: they can be made to converge at points, converge uniformly over an interval, or converge in the mean square sense (which is a pretty close approximation to the uniform sense when the function is smooth and has continuous derivatives). Each of these convergence types is significant in itself and is sufficient to make the Fourier series usable in the context where it converges. Whether they converge at a point, uniformly, or in mean square, the series can be interpreted as an approximation of the function.[14][15]

The pointwise limit functions for converging sequences of functions are straightforward. However, the pointwise sum of an approximating series is a poor substitute for what nearly all of us expect from a Taylor series: accurate approximations of the function. In our context, accurate sums come from using Fourier converging series and, more particularly, mean square convergent series that yield approximations to π, the half period h, and the amplitudes of ascending half sine series approximations of a function. Parseval's identity, which can also be thought of as the (application of the) quadratic law of cosines in the context of the unit circle, is necessary for understanding what sorts of convergences to expect for different kinds of series of functions. Robust and reliable function approximations can be obtained using the Fourier series if they are ensured to converge in the pointwise, uniform, or mean square sense. This foundational knowledge is essential for a variety of problems in analysis and applied mathematics. Parseval's identity and Fourier series convergence have practical impacts that can reach well beyond theoretical mathematics, with influences felt in the domains of engineering, physics, and other scientific disciplines.

2.1. Visual Aids

Graphic displays, such as graphs that demonstrate sine and cosine actions, are very beneficial for better comprehension. The Fourier Series can be used to show how individual sine and cosine values can be combined to reconstruct the same sound waveform if you overlay a diagram of an intricate wave pattern. This visualization illustrates how Fourier analysis is used for sound synthesis in real-world situations as well as helping to understand it.

2.2. Application in Sound Analysis

The Fourier Series gives the mathematical representation for the analysis of waves of Sound waves in particular. Researchers can further use the techniques to decode the frequencies of produced sound as a means to analyze how music instruments produce sound. This analysis is especially useful in recent categorization of musical tones into its first bass and its overtones which prove useful in upward development in such aspects like sound engineering and so on.

Example: Showing the way of Fourier Series in analyzing waves of any signal.

$$f(x) = \begin{cases} -2 & if \quad -\pi < x < 0 \\ 2 & if \quad 0 < x < \pi \end{cases} \quad and \quad f(x+2\pi) = f(x)$$

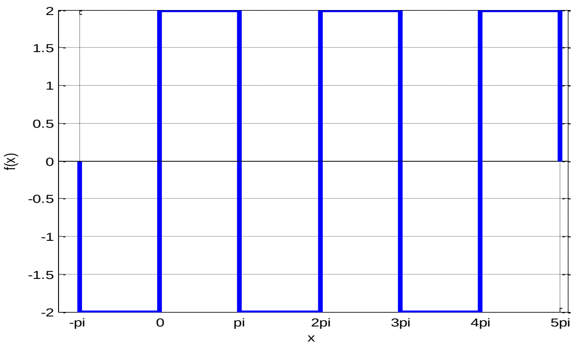

Figure 1: Simple periodic waveform of a signal

By using the Formulas of Fourier coefficients, we get

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kx)\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0}(-2)\cos(kx)\,dx + \int_{0}^{\pi}(2)\cos(kx)\,dx\right]$$

$$\Rightarrow a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kx)\,dx$$

$$= \frac{1}{\pi}\left[(-2)\frac{\sin(kx)}{k}\Big|_{-\pi}^{0} + (2)\frac{\sin(kx)}{k}\Big|_{0}^{\pi}\right]$$

$$\Rightarrow a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kx)\,dx = 0$$

$$\therefore a_k = 0 \;, \forall\, k = 0,1,2,\ldots$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(kx)\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0}(-2)\sin(kx)\,dx + \int_{0}^{\pi}(2)\sin(kx)\,dx\right]$$

$$\Rightarrow b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(kx)\,dx$$

$$= \frac{1}{\pi}\left[(2)\frac{\cos(kx)}{k}\Big|_{-\pi}^{0} + (-2)\frac{\cos(kx)}{k}\Big|_{0}^{\pi}\right]$$

$$\Rightarrow b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(kx)\,dx$$

$$= \frac{2}{k\pi}(\cos(0) - \cos(-kx) - \cos(kx) + \cos(0))$$

$$\Rightarrow b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(kx)\,dx = \frac{4}{k\pi}(1 - \cos(k\pi))$$

$$\because \cos(k\pi) = (-1)^k$$

$$= \begin{cases} \frac{8}{k\pi} if \quad k \quad is\ odd \\ 0 \quad if \quad k \quad is\ even \end{cases}$$

So, the Fourier Series for the given function is

$$f(x) = \frac{8}{\pi}\sin(x) + \frac{8}{3\pi}\sin(3x) + \frac{8}{5\pi}\sin(5x) + \cdots$$

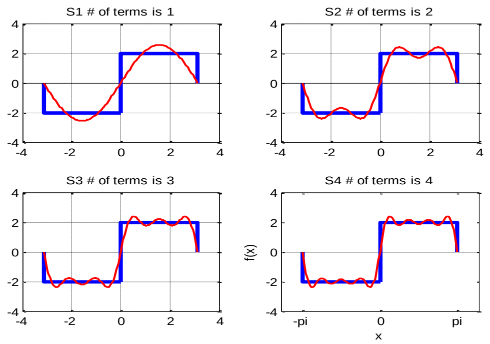

Figure 2: Partial sums of Fourier Series of the given function

3. Practical Applications

Fourier Series have a central position in many domains of music production and acoustics. Through the breaking down of complex sounds waves into sinusoidal elements it improves on the analysis, synthesis and manipulation of sound. Based on above analysis, there are several music industry applications and Fourier Series case studies listed as follows.

3.1. Acoustic Inspection of the Tone and Tuning of Musical Instruments

It is an important application of Fourier Series in real world where sound analysis is concerned especially in instrument tuning and identification. For instance, Fourier writing by researchers has been applied in analyzing what instrument players like violinists or cellos make as overtones. From harmonic series, they should be in a position to understand how different materials used and construction method affect the tone quality of the instrument.

One example is the restoration of the famous Stradivarius violin; Fourier analysis was used to determine the sound frequencies produced from which harmonics belong. It also leads to the providence of historical instruments while also guiding today’s luthiers helping them construct violins with such exceptional aspects.

3.2. This is true for Digital Audio Workstations (DAWs).

Fourier Series, I did find out, is the backbone of Digital Audio Workstations (DAWs), including Ableton Live, Pro Tools, and Logic Pro. Such SW platforms use Fast Fourier Transform (FFT) algorithms which help determine the data of audio signals in real time. For instance, when using equalization, a sound engineer is simply adjusting the levels of certain frequency bands obtained as a result of the Fourier analysis of a certain audio material. When employing a waveform in a DAW, an engineer is guaranteed immediate understanding of the frequency range and therefore has control over the completion of management of specific frequencies as well as their boosting or cutting. This application of Fourier Series has been a great breakthrough in music production since it has provided producers with professional quality work instigate.

3.3. Sound Synthesis Techniques

Additive synthesis and frequency modulation synthesis which are used in synthesis of sound rely on Fourier Series. In additive synthesis, sources of sound waves are produced from a single point only, and different sine waves are then added together to create other sounds, further complicating its timbres. This method is especially widely used in synthesizers such as Yamaha DX7, where such synthesis type generates a great variety of sounds by modulating the frequency of one oscillator with the help of another one.

A real-life example is reproduction of various instrument sounds in today's Popular electronic music. The individual harmonic sine wave utilizing control enables producers to compose two or more sine waves with dissimilar harmonics to achieve the desired textural quality. This capability enables artists to create new sounds which allow productions to venture out from standard music.

3.4. Acoustic Enhancement and Sound Denoising

Fast Fourier Transform helps in noise reduction and audios restoration processes as well. During post-recording, engineers also come across noises, normally interference, which may have a negative impact on the entire quality of a particular data. They also noted that by using such filters, based on Fourier analysis they can effectively delete these undesirable frequencies while not harming the rest of the audio recording.

For example, there is a program called iZotope RX that performs Fourier analysis to display and treat audio spectra enabling one to eliminate noises, clicks, etc in particular parts from the rest. They enhance the listening as well as retain the initial aesthetics of creations that are preserved in the application.

3.5. Real-Time Sound Manipulation

For the physically involved performances, Fourier Series allows real time sound control through a set of effects and processing. For example, and which is very much live electronic musicians, who work mainly with grain synthesis in which sound is divided into grains and then combined in a different way to form new textures and rhythm. This technique actually hinges largely on Fourier analysis in order to properly handle each grain as well as the related frequencies. An example is witnessed in shows where artists such as Amon Tobin who uses real-time sound control to produce audio-visual display. Tobin can thereby build meaningful sounds and layers by using Fourier-based techniques, to create interactive and interesting sound scopes.[16]

4. The Importance of Fourier Series in Music Theory

Music theory and the broader field of acoustics have greatly benefited from the study of Fourier and its series. These developments have provided a robust mathematical framework for understanding and effectively manipulating the kinds of tones musicians make. More specifically, they've given a penetrating insight into that which makes a musical signal what it is.

1) Signals in Music

At its most basic, a musical sound is a signal. This signal has a periodic nature that can be described in terms of frequency, which is the rate at which a cyclic waveform repeats itself. This rate dictates the pitch of the sound. The relationship between frequency and pitch can be expressed as:

$$f = \frac{1}{T}$$

where (f) is the frequency and (T) is the period of the sound wave.

2) Harmonic Composition:

The sound made by a musical instrument, when analyzed, is seen to comprise the fundamental frequency, along with the harmonics (whole-number multiples of the fundamental frequency). The specific blend of these frequencies gives each instrument its own unique tone quality or timbre. The relationship between the fundamental frequency (f_0) and its harmonics can be expressed as:

$$f_n = nf_0$$

where (f_n) is the frequency of the (n)-th harmonic.

4.1. Practical Applications in Music Theory

Using Fourier analysis, we can take apart the waveforms that instruments produce and reconstruct them into the fundamental sine and cosine components that make up the waveforms. This gives us insight into what makes each instrument distinctly "them," even in an ensemble situation where the tonal quality of each instrument is critical to the overall experience of the piece being performed. Another insight gained from analyzing waveforms with Fourier components is how changing the properties of the instrument will alter the output. For instance, if you change the intensity or phase of certain harmonics, the change might be subtle, but it will be perceived as "different" somehow. We can use Fourier analysis to determine the harmonics that contribute to the formation of a musical signal's envelope. This is vital to tasks such as sound compression, noise reduction, and the enhancement of audio signals.

4.2. An Example of the Fourier Method with a Basic Waveform

In order to clarify the using of the Fourier method, let the considered waveform represents a sound note. In other words, any waveform– sine wave, triangular wave, a square-wave, etc. can be constructed from a set of frequencies and harmonics using Fourier series. The same analysis in In Figure 3, we apply a periodic waveform to Matlab, using the following functions:

The linspace function creates linearly spaced vectors that are used to create the time vector.

Waveforms that are produced by sawtooth functions.

- `trapz`: Implemented trapezoidal rule to approximate that definite numerical integration.
- Plot: Used to plot waveforms.
- `legend`, `xlabel`, `ylabel`, `title`, `grid`, `set`, `axis`, `box`: There are, of course, many other plotting functions to help make it look this way: This code is well suited to illustrate a Fourier series for approximating a waveform by summing sine and cosine functions that carry frequency-related information in signals that are processed and analyzed spectroscopically. For computational reasons the Fourier coefficients calculated as follows:

$$\left[a_0 = \frac{1}{T}\int_0^T f(t)\,dt\right]\left[a_n = \frac{2}{T}\int_0^T f(t)\cos\left(\frac{2\pi nt}{T}\right)dt\right]\left[b_n = \frac{2}{T}\int_0^T f(t)\sin\left(\frac{2\pi nt}{T}\right)dt\right]$$

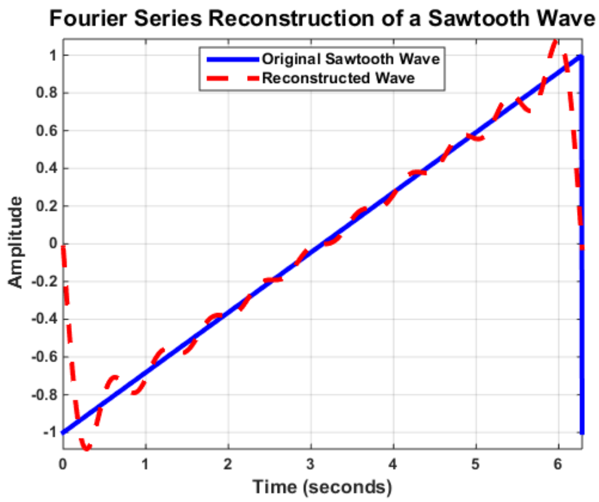

Figure 3: Reconstructed the waveform using Fourier series

This is true because Fourier series keeps a lot of help in waveform analyses and even in making some changes on it. In a way, they rather disassemble a sound to reveal its harmony beneath. In this manner we are able to forecast and influence the undertones that an instrument creates when it is played. It is widely used in the mathematics of music and especially in the construction of sound wave forms or indeed any application of signal processing mathematics.

5. Basics of Music Theory

This section outlines the methodology employed to explore the foundational concepts of music theory, specifically focusing on the characteristics of octaves, intervals, and scales. The primary aim is to provide a comprehensive understanding of how these concepts interrelate and contribute to the overall framework of music theory. Through a combination of theoretical analysis, practical applications, and empirical observations, we aim to illustrate the core principles that underlie both harmonic structures and the science of acoustics.

5.1. Musical Scales

A musical scale can be thought of as a selection of certain notes out of the octave to form a pattern to which melodies and harmonies conform to. Western music has one predominant scale called the major scale where you play the seven notes at regular intervals apart from each other.

5.1.1. Musical Scales' Systems

By changing the tuning system profoundly, attitudes toward musical intervals become entirely different. The two most prevalent systems are:

1. Equal Temperament: This puts the octave into 12 equal semitones meaning there is a way any instrument can play particularly in each key. It makes tuning easier and helps to modulate but a little degrades the harmonic quality of intervals. Equal temperament provides consistent conditions for the representation of harmonically linked waves as each note has a fixed frequency in Fourier analysis.
2. Just Intonation: Unlike this system, the using of whole number ratios for determining intervals creates purer chords. However, there are limitations with modulation because tuning of the bass is key specific. In Fourier analysis, just intonation introduces another approach to the interpretation of the relations between different harmonics, because of higher densities, yields more resonant sounds.

5.1.2. Different Musical Scales

1. Equal Temperament Scale: Equal temperament scale is a system where an octave divides an equal number of 12 parts called as semitone. In this scale, the frequency ratio between two adjacent notes (N and N[+1]) is approximately 1:1.0595. It enables pitch and can be proved to be tuned to this extent that it plays in any musical key and as such is a very important aspect of the western music. The relationship between notes and their frequencies can be expressed as:

$$f_n = f_0 \times r^n$$

where $(r)$is the frequency ratio for the scale, and (n) is the number of steps from the base frequency.

2. Constructing Scales; Specifically, music scales are created through choosing of notes available in one or the other octave range. This mean that each scale is different most significantly due to the fact that the intervals between the notes of the certain scale are different. They are essential to the character of scale and everything related to its musicality.
3. Ratio and Frequency: This means that not only does it happens that scales are built as they do, but distances between notes also correspond to the frequencies of notes. A more harmonious ratio helps create a more consonant sound when the notes under consideration are requited. For example, the interval between C and E in the context of a C major scale is consonant because these two notes have a frequency ratio of 4:5.

5.2. Application of Fourier Series in Music Theory

All these musical concepts are supported mathematically by Fourier series. By breaking a musical note into tones also, it gives easier interfaces for musicians and music theorists, to work with these basics by visualization. For example, when you use Fourier analysis of recording, one can observe that harmonic overtone matches with intervals and scales which are used in composition. Moreover, the utilization of Fourier Series in understanding these relationships can stimulate experimental elements of compositional strategies, through tuning or scale, as a means of achieving a specific sound.

5.3. Characteristics of Octaves

The study utilizes a mixed-methods design, integrating both qualitative and quantitative approaches to analyze the principles of music theory systematically. The qualitative aspect encompasses a literature review of existing music theory texts and academic articles, while the quantitative component involves practical experiments and data analysis related to the concepts of octaves, intervals, and scales.

Octaves are clearly separated by a fixed distance, but being octaves, they do not matter what notes are between them (from Do to Re, for example). The D note too is present aligned through octave (i.e., D3) in the same frequency order as noted above. That is how the count of semitones up from note C0 gives 62 MIDI note number for D3. The relationship can be expressed as:

$$f_{octave} = 2f_0$$

This means if (f_0) is the frequency of a note, (f_{octave}) is the frequency of the same note one octave higher. The frequencies of notes within an octave can be represented in a Table 1:

Table 1: Frequencies of notes in one octave (MIDI Standard)

Note	Frequency
'C4'	261.63
'C#4/Db4'	277.18
'D4'	293.66
'D#4/Eb4'	311.13
'E4'	329.63
'F4'	349.23
'F#4/Gb4'	369.99
'G4'	392
G#4/Ab4'	415.3
'A4'	440
'A#4/Bb4'	466.16
'B4'	493.88

The frequency of D3 is approximately 293.6648 Hz.

Figure 4, show that This loop runs through the number of harmonics which set the amount of variation within the waveform. Left harmonics only are there, sometimes referred to as odd harmonics hence discussing $1, 3, 5$ ... etc and in terms of '`$n = 2 * k - 1$`. Harmonic takes the Fourier series formula and uses it to calc-u-late each of the harmonic components in the square wave. Each harmonic component is accumulated into `reconstructed Signal`. The Figure 4, show how a square wave can be reconstructed from its Fourier series components implying that only odd harmonics are necessary when reconstructing non-sinusoidal waveforms.

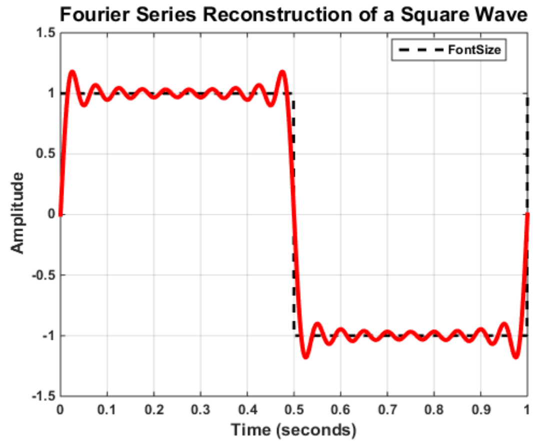

Figure 4: Fourier Series econstruction of a Square Wave

5.4. Intervals and Semitones

1.The Spaces Between Notes in an Octave: Between the notes in an octave there are gaps which are filled what is known as semitones, sometimes referred to as a half step. Every semitone is related to a given frequency ratio.

2. The Value of the Twelfth Root of 2: The ratio between the consecutive semitone is equal to the twelfth root of 2, approximately equal to (1.05946). This is crucial for constructing scales and understanding musical intervals:

$$f_{n+1} = f_n \times 2^{1/12}$$

It is done in such a way that to hit twelve semitones – the layout of one octave, the frequency of blinking doubles.

In Figure 3 a MATLAB code to respectively calculate and then plot out the frequencies of musical notes in one octave using the equal temperament tuning of note A4(440Hz).

`A4` is an element into which frequencies of the musical note A4, 440 Hz, are placed.

num semitones_ is the number of semitones in an octave; this number is 12.

`ratio` is the frequency ratio between two purchased order methods of consecutive notes expressed by equal temperament which is equal to twelfth root of 2, and it means that each Semitone will be equal on a logarithmic equation.

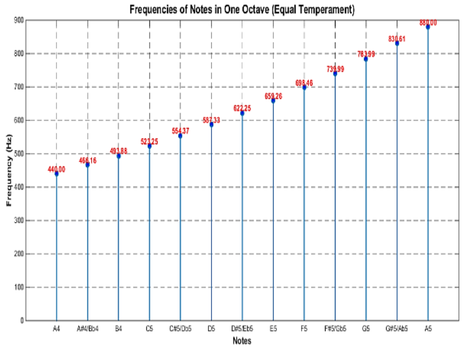

Figure 5: Frequencies of Notes in One Octave (MIDI Standard)

Figure 5 also outlines weights of notes that spread one octave based on the MIDI standard. A number of musical notes and their frequency have been illustrated on the above diagram. The interval is defined as the distance of two notes which cause change in pitch and is the area between two adjoining notes. They point out that this difference is fixed and is equal to the twelfth root of the number 2, or approximately 1.05946 by which the frequency of one note has to be multiplied to get the next one. This factor is vital when giving explanations concerning the octave in music theory.

6. Fourier Method

Fourier method is an analytical powerful technique for analysis and reconstruction of waveform. In this paper, through MATLAB simulation, we are going to show the application of Fourier Series in synthesizing a waveform such as a square wave. In the next segments, the steps are explained as well as their PHP code examples are given.

6.1. Generating a Basic Waveform

The first step and which is very simple involves generating a basic or a simple square wave using MATLAB. It worth to notice that the described square wave can be represented using Fourier Series by applying the series of sine waves. Here's how to generate and visualize this waveform:

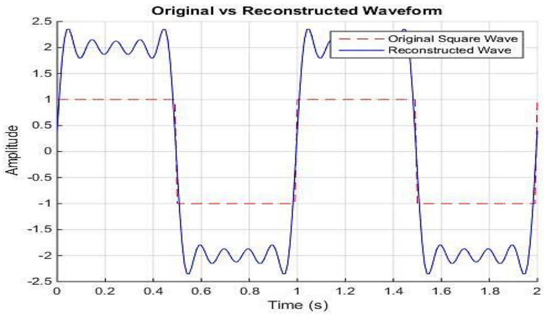

Figure 6: Plot the original and reconstructed waveforms

- Parameters Section: In defining the period of the square wave and the sampling frequency it is represented as (T). To account for an estimate of two real cycles of the wave, the time vector (t) is constructed.

- Square Wave Generation: The `square` function produces a waveform that has a square wave having a given quart.

The existence of both a phase slave and tripod on the rover is an indication that multiple IRCAs were to be onboard the Spirit rover, although the design for a cabin-mounted IRCAs was scaled back to only one and integrated into a tripod-like structure.

- Plotting: The waveform is plotted against time with the help of plot function and the resultant waveform depict high and low value swap pattern of square waveform.

6.2. It is the Fourier Series Coefficients Calculation.

The next thing we are going to do involves the Fourier Series coefficients of the square wave. This involves determining the average value, cosine coefficients, and sine coefficients:

-Initialization: With these assumptions, the sequence we would generate will start with the average value; a_0.

- Waveform Reconstruction: The loop of reconstruction adds the partial sum of Cosine and Sine part till the total wave is reconstructed for all harmonics.

- Visualization: The resulting both the square wave signal and the reconstructed waveform are shown, so it is possible compare and observe how close the reconstruction is.

This example shows the use of Fourier series in analysis and synthesis of a waveform through MATLAB. In Fourier analysis, a square wave is decomposed into Fourier coefficients and then rebuilt Understanding how this is done gives a more profound understanding of how a complicated sound can be mathematically represented. It may be further generalized to other kinds of waveforms and has substantial applications in the synthesis and processing of musical sounds.

6.3. Basic Waveforms

1. Sine Wave:

The simplest form of a waveform, represented mathematically as:

$$f(t) = A\sin(2\pi f t + \phi)$$

where:

- (A) is the amplitude,
- (f) is the frequency,

The following symbols holds true: - (ϕ) is the phase shift.

Sine waves are used to construct simple waves of sound and are used often to mimic actual pure tones.

2. Square Wave:

A square wave can be synthesized using its Fourier Series representation:

$$f(t) = \frac{4A}{\pi}\sum_{n=1,3,5}^{\infty}\frac{1}{n}\sin(2\pi n t)$$

From this representation it will be observed that a square wave indeed represents an infinite sum of sine waves odd harmonics only though each of these sine waves are of diminishing amplitude.

3. Triangular Wave:

The Fourier Series representation for a triangular wave is given by:

$$f(t) = \frac{A}{2} - \frac{A}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\cos(2\pi n t)$$

Like the square wave, the triangular wave consists of harmonics However, the coefficients tail off faster which makes for a smoother sound.

6.4. Synthesizing the Waveforms

For these waveforms, if we have to implement the MATLAB, it is possible to estimate their Fourier coefficients and then sum them to get the waveforms. Below is an example MATLAB code snippet that demonstrates this process for a square wave: Figure 7 shows a MATLAB code that reconstructs a Fourier series from the original square wave. A graphic is then displayed over the raw and reconstructed signals to illustrate the results. As a first step, we must set the size parameters of the reconstructed wave to zero. In order to be a time vector, it must be the same size as the time vector `t`. While displaying Fourier series for square waves, this approximation will only display odd harmonics. In addition, the original result and the reconstruction are directly compared.

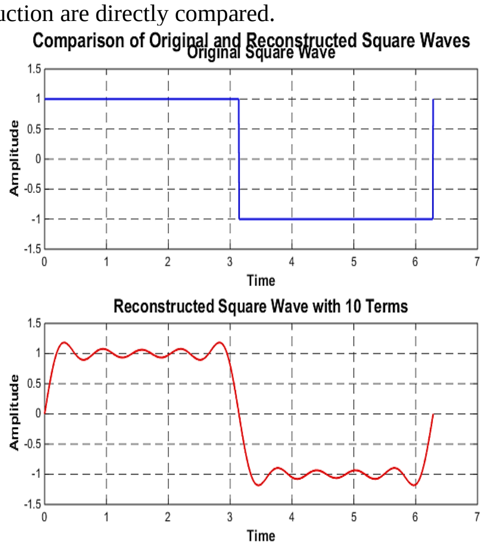

Figure 7: Original and Reconstructed Waveform

7. Implications

The uses of Fourier analysis in music today have made major changes in several aspects of sound technology but there are still future possibilities. Promising areas for future growth suggest several themes that could potentially alter the manner in which we produce, alter, and interact with sound in the future of research.

7.1. Recent Developments in Sound Generation

Future expansion of this function may use Fourier analysis at a higher level in more precise sound synthesis. With improved computational power it may be possible to find new synthesis techniques which mirror the sound producing characteristics of real musical instruments. For example, the integration of additive, subtractive and physical modeling could produce high density and complex signals which would come closer to the actual acoustic instruments.

7.2. Improved or Advanced Algorithms of Voice and Sound Processing

As continued work on applying Fourier analysis is still under way, we can hope for the development of better techniques for processing audio signals. This improvement can result in better live audio processing options like real-time equalizer, low-frequency noise elimination, surrounding sound distribution and so on. For instance, the machine learning algorithms coupled with Fourier analysis can be used to include aspects of audio processing that would depend on particular performance or recording and apply adjustments to sound in real time.

7.3. Immersive Audio Experiences

Since VR and AR technologies are developing year by year, the application of Fourier analysis is letting people feel real sound will be more essential. This paper provides information about how Fourier techniques can be used to improve sound in virtual reality applications. It could result in stronger engagement of the user and more realistic surroundings where sound responds explicitly to their actions.

7.4. Algorithmic Composition and Artificial Intelligence

Fourier analysis followed by application of artificial intelligence can be a breakthrough for algorithmic composition system. Using Fourier techniques, computer-based AI systems could analyze myriad databases containing music, so as to determine the patterns and structures that characterize different forms of music. Such information could be applied in the generation of other compositions which should bear all the characteristics of the current compositions but contain elements of novelty. Such interactions between AI and human expert could bring in new potential music styles and form which are yet to be developed.

7.5. Metaphysical Use

The concept of Fourier analysis in music is therefore not restricted to sound technology only. In future studies, there would be interesting applications to areas like neuroscience, where Fourier methods of assessing the cognitive processing of music can improve therapies of patients with auditory processing disorders. Moreover, conclusions drawn from Fourier analysis could be applied to advances in acoustical engineering to enhance sound quality in plazas, concert halls, and studios.

7.6. Education and Accessibility

At last, more significantly, provided that Fourier analysis plays a more important role in music technology, helpful educational aids may be created to enhance students' understanding of these theories and principles. It is mentioned, to make Fourier transforms more comprehensible for students and other aspiring musicians, there are specific interactive routines and applications out there for this kind of use. Such democratization of knowledge can trigger new generation of musicians and sound engineers to create more innovation in the production of music.

7.7. MATLAB Application

This process can be computationally solved in MATLAB since this tool has reliable Fourier coefficients. The coefficients can be computed using the following MATLAB code snippets:

1. Calculating the Fourier Coefficients:

```matlab
T = 1; % Period
f = @(t) ... ; % Define your waveform function here
a0 = (1/T) * integral(f, 0, T);
an = @(n) (2/T) * integral(@(t) f(t).*cos(2*pi*n*t/T), 0, T);
bn = @(n) (2/T) * integral(@(t) f(t).*sin(2*pi*n*t/T), 0, T);
```

2. Reconstructing the Waveform:

```matlab
t = linspace(0, T, 1000); % Time vector
f_reconstructed = a0/2; % Start with a0/2
  for n = 1:N % N is the number of harmonics
f_reconstructed = f_reconstructed + an(n)*cos(2*pi*n*t/T) + bn(n)*sin(2*pi*n*t/T);
  end
plot(t, f_reconstructed); % Plot the reconstructed waveform
```

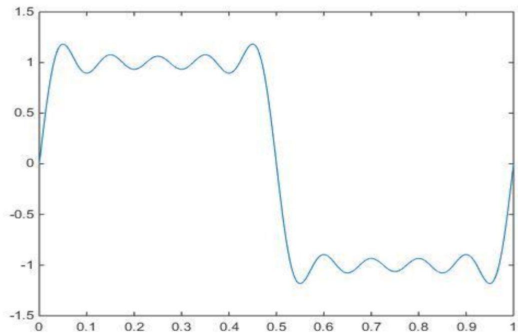

Figure 8: Plot the reconstructed waveform

To examine the characteristics of the D note across different octaves, we focused on the following frequencies, as illustrated in Table 2:

Table 2 : Frequencies of the D Note Across Different Octaves

Octave	Frequency Hz
Second Octave	146.83
Third Octave	293.66
Fourth Octave	587.33
Fifth Octave	1174.7

Additionally, as shown in Figure 9, MATLAB code is used to generate and plot sine waves for a musical note (D) across different octaves. A visual representation of how the note's frequency changes as it is played at higher octaves can be found in the code below. In music theory and signal processing, it is fundamental to understand how the frequency of notes changes across different octaves.

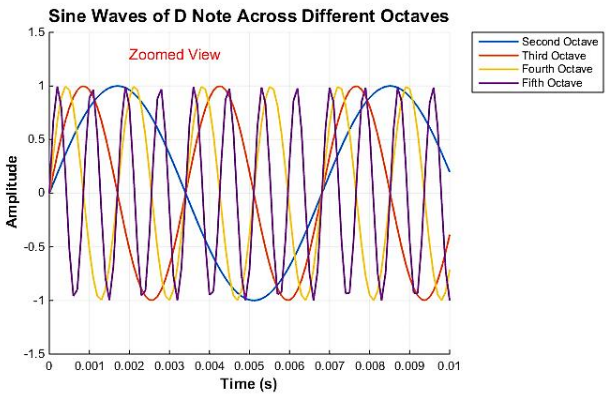

Figure 9: Sine Wave of D Note Across Different Octaves

With these MATLAB patterns, researchers and sound engineer can work much better in analyzing the musical tones and synthesizing them for better understanding in sound producing.

8. Results

Applying Fourier Series to music analysis reveals several key insights that enrich our understanding of musical complexity. The main contributions of this research include:

- **Decomposition of Sound Waves**: Fourier Series enables the breakdown of complex sound waves into fundamental sine and cosine components. This decomposition helps represent musical notes and harmonies as combinations of these basic waveforms, providing a detailed view of their structure.
- **Harmonic Analysis**: Fourier analysis isolates musical tones into their primary frequency components and harmonic overtones, offering insights critical for studying the acoustic characteristics of musical instruments.

- **Applications in Sound Synthesis**: Fourier Series underpins various sound synthesis techniques, including additive synthesis and frequency modulation, by combining fundamental waveforms to create diverse timbres.
- **Real-Time Audio Processing**: Fourier analysis enhances real-time audio processing capabilities in modern Digital Audio Workstation (DAW) software, allowing precise control and fine-tuning of frequency elements during music production.
- **Impact on Music Theory**: Understanding the mathematical basis of Fourier Series broadens perspectives in music theory, equipping composers and musicians with tools to analyze harmonic structures, experimental scales, and tuning systems.
- **Cross-Disciplinary Insights**: This study demonstrates that Fourier analysis extends beyond music, impacting fields like acoustics, audio engineering, and cognitive neuroscience by providing foundational methods for analyzing sound.
- **Potential for Future Innovations**: Advancements in Fourier analysis techniques hold promise for new developments in signal synthesis, immersive audio applications, and the integration of artificial intelligence in music composition.

Together, these findings illustrate how Fourier Series has transformed traditional music theory, integrating it into modern sound engineering and paving the way for musical innovation.

9. Conclusion

This investigation underscores the profound relationship between mathematics and music, showcasing Fourier Series as a versatile tool for music synthesis, analysis, and performance. The applications of Fourier analysis extend beyond classical music production into areas like sound design, archaeology, cognitive neuroscience, and audio engineering. By breaking down musical waves into simple sinusoidal components, Fourier Series enables musicians and sound engineers to analyze and reconstruct sounds, designing new and innovative auditory experiences.

As exploration continues, Fourier analysis is likely to yield new insights in both sound production and musical analysis.

10. Future Research Directions

To further harness the potential of Fourier analysis in music, the following areas warrant investigation:

1. **Enhanced Sound Synthesis**: Future research could focus on refining synthesis algorithms through Fourier techniques, leading to more realistic and complex synthesized sounds.
2. **Machine Learning Integration**: Combining Fourier analysis with machine learning may unlock applications in automated music composition and real-time sound processing, opening new avenues for creative expression.
3. **Cognitive Processing of Music**: Applying Fourier analysis to study auditory perception could improve understanding of how the brain processes musical sounds and provide insights for music therapy.
4. **Cross-Genre Applications**: Investigating Fourier analysis across different music genres and cultural scales could reveal insights into the tuning systems and harmonic structures unique to each tradition.
5. **Educational Tools**: Real-time 3D applications simulating Fourier transformations in music could enhance educational tools, offering students and beginners engaging, interactive ways to grasp mathematical concepts in music.

Thus, the connections established between mathematics and music through Fourier Series offer not only intellectual enrichment but also opportunities for innovative educational and creative applications. By continuing to study these relationships, we open up possibilities for technological and artistic advancements in the field of music.

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