Some Properties of Fuzzy Compact topological Space

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Abstract Fuzzy topological space is said to be a fuzzy compact if every fuzzy open cover has finite sub cover. The aim of this paper is to use fuzzy finite intersection property to prove some properties of fuzzy compactness. Also, a continues image of fuzzy compact space is fuzzy compact is proven. some numerical examples are given to illustrate the results.

Keywords: Fuzzy set; Fuzzy Ffinite Intersection Property; Fuzzy Topology Space; Fuzzy Compact; Fuzzy Continuity.

بعض الخصائص للفضاء الطوبولوجي المتراص ضبابيا

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الملخص نقول ان الفضاء الطوبولوجي الضبابي يكون متراص ضبابيا ادا كان كل غطاء مفتوح ضبابي يمكن استخلاص منه غطاء ضبابي جزئي منتهي. الهدف من هده الورقة تقديم فكرة جديدة لا تبات ان الفضاء الضبابي يكون متراص ضبابيا ادا وفقط ادا كان لاي عائلة من المجموعات الجزئية المغلقة ضبابيا لها خاصية التقاطعات المنتهية كدلك الصورة المستمرة لفضاء ضبابي متراص تكون متراصة ضبابيا. وتم إعطاء بعض الأمثلة التي توضح النتائج.

الكلمات المفتاحية: المجموعة الضبابية، خاصية التقاطعات المنتهية الضبابية، الفضاء الطوبولوجي الضبابي، التراص الضبابي الاستمرارية في الفضاء الطوبولوجي والفضاء الطوبولوجي الضبابي.

1. Introduction

Fuzzy set theory was discovered by Zadeh [1] in 1965. Three years later, Fuzzy topology is introduced by Chang [2]. He gave the concept of compact fuzzy topological spaces were first introduced in the literature. Two results about such spaces were proven, in terms of open covers. This compactness is defined for any fuzzy subset and has more advantages. compactness theory by showing that the Tycho off Theorem is false for infinite products. Moreover, Gartner et al introduced several researches were conducted on the generalizations of the notions of fuzzy sets and fuzzy topology. As a result, it gradually has been widely accepted. Based on this, a series of works. Loewen [5] introduced fuzzy compactness strong fuzzy compactness and ultra-fuzzy compactness, Liu introduced Q-compactness [12,13] Wang and Zhao introduced N-compactness [10], Although Wong [8] treats compactness, his results are not significant. Loewen [5] gives a different definition of a compact fuzzy space and drastically alters the definition of a fuzzy topological space the usual definition of a fuzzy topology includes ordinary topologies, but Loewen's definition of a fuzzy topology excludes ordinary topologies, from being fuzzy topologies and the aim to introduce a new notion of proof that thermos of fuzzy compactness in fuzzy topological spaces .In 2019 present a softer-partition notion is presented and point out this notion is sufficient for the equivalent among the four types of soft pre-compact spaces and for the equivalent among the four types of soft pre-Lindell of spaces. They demonstrate the relationships between enriched soft topological spaces and the initiated spaces in different cases and obtain interesting results. In this work, we use the fuzzy finite intersection property as a necessary and sufficient condition to prove the fuzzy compactness. In addition, we show that the fuzzy compactness is satisfied under fuzzy Continuity.

2. Preliminaries

In this section, we shall describe or, we Firstly, present the fundamental definitions of fuzzy set fuzzy topological space and fuzzy compact space.

Definition 2.1:

Let X be a non-empty set, a collection T of subsets of $X . T = \{A : A \subseteq X\}$ is said to be a topology on X if

(i) $X \in T, \emptyset \in T$

(ii) If $A\_i \in T , \forall i \in I$ then $\cup A\_i \in T$

(iii) IF $A\_i \in T, i = 1 \dots n$ then $\cap A\_i \in T$ **[3].**

Definition 22.2:

Let $(X , T_1), (Y , T_2)$ be two topological spaces, and f a function from X to Y. Then we say that

a) f is continuous iff for every open set B in T_2, we have $f^{-1}(B)$ is open set in T

b) f is open iff for any open set A in T, the set $f(A)$ is open set in T_2 **[20]**

Definition 2.4:

Let X be a non-empty set and let I be the unit interval $(i.e., I = [0,1])$. A fuzzy set A in X is a function from X into the unit interval I.

$$i.e., \mu_A : X \to [0,1].$$

A fuzzy set A in X can be represented by the set of pairs,

$$A = \{(x, \mu_A(x)) : x \in X\}$$

The family of all fuzzy sets in X is denoted by I^X **[15]**

Definition 2.4:
Let $A = \{(x, \mu_A (x) : x \in X\}$ and $B = \{(x, \mu_B (x) : x \in X\}$ be two fuzzy sets in X. Then their union $A \cup B$, intersection $A \cap B$ and complement A^C are also fuzzy sets with the membership functions defined as follows:

(i) $\mu_{(A \cup B)} (x) = max \{\mu_A(x), \mu_B (x)\}, \forall x \in X.$
(ii) $\mu_{(A \cap B)} (x) = min \{\mu_A(x), \mu_B (x)\}, \forall x \in X.$
(iii) $\mu_{A^c} (x) = 1 - \mu_A(x), \forall x \in X.$
 Further
(a) $A \subseteq B$ iff $\mu_A (x) \leq \mu_B (x), \forall x \in X.$
(b) $A = B$ iff $\mu_A (x) = \mu_B (x), \forall x \in X$ **[16]**.

Definition 2.5:
A fuzzy topology on a set X is a collection T of fuzzy sets in X satisfying:
(i) $0 \in T$ and $1 \in T$
(ii) If A and B belong to T, then $A \wedge B \in T$
(iii) if A_i belongs to T for each $i \in J$ then so does $\vee_{i \in J} A_i$.
Here, $A \wedge B$ and $\vee A_i'$ are defined by
$$(A \wedge B)(x) = inf\{A(x), B(x)\}$$
$$\vee A_i (x) = sup\{A_i (x): i \in J\}, \text{for } x \in X.$$
If T is a fuzzy topological on X, then the pair (X, T) is called a fuzzy topological space. Member of T are called fuzzy open sets. fuzzy sets of the forms $1 - A$, where A is fuzzy open set are called fuzzy closed sets. **[17]**

Theorem 2.1:
Let $A \leq Y \leq X$ .Then:
(i) If A is a fuzzy open set in Y and Y is a fuzzy open set in X, then A is a fuzzy open set in X. (ii) If A is a fuzzy closed set in Y and Y is a fuzzy closed set in , then A is a fuzzy closed set in X **[18]**

Definition 2.6:
Let X and Y be two non–empty sets $f: X \to Y$ be function. For a fuzzy set B in Y, the inverse image of B under f is the fuzzy set $f^{-1} (B)$ in X with membership function denoted by the rule:
$f^{-1} (B)(x) = B(f(x))$ for $x \in X$ $(i.e., f^{-1} (B) = Bof)$
For a fuzzy set A in X, the image of A under f is the fuzzy set $f(A)$ in Y with membership function $f(A)(y), y \in Y$ defined by
$$f(A)(y) = \begin{cases} sup_{x \in f^{-1}(y)} A(x) & if \ f^{-1}(y) \neq 0 \\ 0 & if \ f^{-1}(y) = 0 \end{cases}$$
Where $f^{-1} (y) = \{x: f(x) = y\}$ **[14]**

Definition 2.7:
Let (X, T_1) and (Y, T_2) be two fts and let
$$f : (X, T_1) \to (Y, T_2)$$
be a mapping. Then f is said to be fuzzy-continuous (f- continuous) if $f^{-1}(B) \in T_1$, For each $B \in T_2$. **[19]**

Definition 2.8:
A family U of fuzzy sets is a cover of fuzzy set A if and only if $A \leq \vee \{B_i: B_i \in U, \forall i \in J\}$. It is called fuzzy open cover if each member B_i is fuzzy open set. A subcover of U is a subfamily of U. which is also a cover of A **[13]**

Definition 2.9:
Let (X, T) be a fuzzy topological space and let $A \in I^X$. Then A is said to be a fuzzy compact set if for every fuzzy open cover of A has a finite fuzzy sub cover of A. If $A = X$, then X is called a fuzzy compact space that is $A_i \in T$ for every $i \in I$ and $\vee A_i = 1, i \in I$, then

there are finitely many indices $i = 1, i\_2, \dots, i\_n \in I$ such that $\vee_{i \in I} A_i = 1$. **[14]**

Theorem 2.2:
A fuzzy topological space (X, T) is fuzzy compact if and only if for every collection $\{A_i, i \in J\}$ of fuzzy closed sets of X having the finite intersection property, $\wedge_{i \in J} A_i \neq 0$. **[14]**

Theorem 2.3:
In any fuzzy space, the intersection of a fuzzy compact set with a fuzzy closed set is fuzzy compact. **[14]**

Theorem 2.4:
A fuzzy closed subset of a fuzzy compact space is fuzzy compact. **[13]**

Theorem 2.5: Let f be a function from a set X into a set Y. If A, $A_i, i \in J$ are fuzzy sets in X and if $B, B_K, K \in I$ are fuzzy sets in Y,
then the following are true,
(i) $f[f^{-1} (B)] = B$, when f is onto.
(ii) $f(\wedge A_i) \leq \wedge f(A_i).$
(iii) $f(\vee A_i) \leq \vee f(A_i).$
(iv) $f^{-1} B_K \leq \wedge f^{-1}(B_k).$
(v) $f^{-1} B_k \leq \vee f^{-1}(B_k).$
(vii) $f[f^{-1}(B) \wedge A] = B \wedge f(A).$ **[19]**

3. Main Results

Proposition 3.1:
A fuzzy space X is fuzzy compact iff any collection of closed fuzzy sub sets of X has the finite intersection property

Proof
Suppose that X is fuzzy compact.
Let $\{F_\propto\}(\propto \in J)$ be any collection of closed fuzzy sets in X such that
$$\wedge_{\propto \in J} F_\propto = 0 \tag{3.1}$$
Then
$$X = 1 - \wedge_{\propto \in J} F_\propto = \vee_{\propto \in J}(1 - F_\propto) \tag{3.2}$$
Therefore
$$\{1 - F_\propto\}, \propto \in J \tag{3.3}$$
is a fuzzy open set of X,
$$X = \vee(1 - F_{\propto_i}), i = 1, 2, \dots \tag{3.4}$$
Since X is a fuzzy compact space. Then there is a finite fuzzy open sub cover
$$X = \vee(1 - F_{\propto_i}), i = 1, 2, \dots, n \tag{3.5}$$

$$X = 1 - \wedge(F_{\propto_i}), i = 1, 2, \dots, n \tag{3.6}$$
then $\wedge F_{\propto_i} = 0, i = 1, 2, \dots, n$.

Conversely
suppose that $\{U_\propto\}(\propto \in J)$ is a fuzzy open cover of X. Then $\{X - U_\propto\} \propto \in J$ is a collection of closed fuzzy sets.
But X has the finite intersection property
$$\text{Then } \wedge(X - U_{\propto_i}) = 0, i = 1, 2, \dots$$
$$X - \vee U_{\propto_i} = 0.$$
$$X = \vee U_{\propto_i},$$
 then X is fuzzy compact.

Example3.1:
Let $X = \{a, b, c\}$ be a fuzzy set.
$$A = \{(a, 0), (b, 0.4), (c, 1)\}$$
$$.F = \{(a, 1), (b, 0.6), (c, 0)\}$$
collection of closed fuzzy set in X such that
$$\wedge F = 0. \ X = 1 - \wedge F = \vee (1 - F).$$
$$X = 1 - \wedge F$$
$$\therefore \wedge F = 0$$

Proposition 3.2:

A continuous image of a fuzzy compact space is fuzzy compact

Proof:

Let $f: X \rightarrow Y$ be continues map and X a fuzzy compact space.

Suppose that $\{U_\propto\}, \propto \in J$ is a fuzzy open set of $f(X)$.

$$f(X) = \vee_{\propto \in J} U_\propto \qquad (3.7)$$

then

$$X = f^{-1} (\vee U_\propto, \propto \in J) \qquad (3.8)$$
$$X = \vee f^{-1} (U_\propto), \propto \in J \qquad (3.9)$$

Therefore, $\{f^{-1}(U_\propto)\}, \propto \in J$ is a fuzzy open cover of X. Then

$$X = \vee f^{-1}\left(U_{\propto_1}, U_{\propto_2}, \dots, U_{\propto_n} \right) \qquad (3.10)$$
$$\therefore f(X) = f(X) = U_{\propto_1} \vee U_{\propto_2} \vee \dots \vee U_{\propto_n} \qquad (3.11)$$

Thus, $f(X)$ is fuzzy compact space.

Example 3.2:

Let $X = \{x_1, x_2, \dots, x_n\}$ is a fuzzy compact. And let mapping $f: X \rightarrow Y$ be an onto continues function. let $U = \{0, Y\}$ is an open cover of Y then

$$Y = \vee U. \therefore X = f^{-1}(\vee U) = \vee f^{-1}(U) \qquad (3.12)$$

f is continues, then $f^{-1}(U)$ is an open cover of X. But X is fuzzy compact then

$$X = f^{-1}(0) \vee f^{-1}(Y). Y = f(X) = 0 \vee Y \quad (3.13)$$

$\therefore Y$ is a fuzzy compact space.

Remark 3.3

Loewen's, Chang…, strong fuzzy compactness and ultra-fuzzy compactness of topological spaces are extended to these notions as well. Introduce a new notion of proof that thermos of fuzzy compactness in fuzzy topological spaces. We recall the definition of fuzzy set, fuzzy topological space and fuzzy compactness.

Conclusions:

Since the fuzzy set allows us to represent fuzzy concepts naturally, we find that the concept of the fuzzy compact topology space. It is the application of fuzzy sets as a generalization of the compact topology, from which we construct and study two generalizations of fuzzy compact topological spaces with the help of illustrative examples and other proofs, While the concepts presented in the paper are necessary for further research and will open the way to improve more applications on the fuzzy topology.

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