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On the Oscillation Property for Some Nonlinear Differential Equations of Second Order

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Abstract In this paper, oscillatory behaviour of solutions of a class of nonlinear second-order ordinary differential equations is investigated. By using Riccati substitution technique, some new sufficient conditions for the oscillation are given. Comparisons between our results and the previously known results are presented. Furthermore, some illustrative examples are also included to show the evidence of the new results. However, results obtained here improve and/or extend some of well known results in the literature.
Keywords: Nonlinear differential equations, Oscillation, Riccati transformation, Second order.

عن خاصية التذبذب لبعض المعادلات التفاضلية الغير خطية من الرتبة الثانية

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الملخص في هذا البحث تمت دراسة سلوك التذبذب لحلول بعض أنواع المعادلات التفاضلية العادية الغير خطية من الرتبة الثانية. تم استخدام تحويل ريكاتي للحصول على شروط كافية للتذبذب. أيضا تمت المقارنة بين النتائج المتحصل عليها في هذا البحث والنتائج المعروفة سابقا. تم الاستدلال ببعض الأمثلة التطبيقية لتوضيح أهمية النتائج المتحصل عليها. النتائج المتحصل عليها في هذا البحث هي تحسين وتوسيع لعدد من النظريات السابقة.

الكلمات المفتاحية: معادلات تفاضلية غير خطية، تذبذب، تحويل ريكاتي، رتبة ثانية.

Introduction

In recent years, the literature on the oscillation theory of nonlinear differential equations is growing very fast following the publication of the work of Atkinson [2]. It is relatively a new field with interesting applications in real world life problems. In fact, many chemical and physical systems are modeled by nonlinear second order differential equations. For example, the study of the Emden-Fowler equation originates from earlier theories concerning gaseous dynamics in astrophysics around the turn of the century. For a discussion on the Emden-Fowler equation, we refer to the paper of Wong [20].

Here, we are concerned with the problem of oscillation of solutions of the following second order nonlinear differential equation

$$\left(r(t)\dot{x}(t)\right)^{\bullet} + q(t)|x(t)|^{\gamma} signx(t) = o, \quad t \geq t_0 \qquad \text{(I)}$$

where q and r are continuous functions on the interval $[t_0, \infty)$, $t_0 \geq t$, $r(t)$ is positive function on the real line $\Re$ and $\gamma > 0$.

Throughout this study, we restrict our attention only to the solution of the differential equation (I) which exists on some interval $[t_0, \infty)$, $t_0 \geq 0$ which may depends on a particular solution. It is worthy to notice that in the study of the oscillation nature of second order differential equations, the technique of integral averaging plays an important role, which goes back to the classical paper of Fite[8]. Also, one can see the paper of Kamenev [12], which involve the averaging criterion and its generalizations. For such averaging techniques related to this area, one can see Butler [5], Grace and Lalli [10], Kwong and Wong [11], Philos [17], Wong [20], Yan [22] and Yeh [23].

A regular solution of Eq. (I) which is defined for all large t is called oscillatory if it has no last zero, otherwise, it is said to be nonoscillatory. Thus a nonoscillatory solution is eventually positive or negative. Consequently, equation (I) is called oscillatory if all its solutions are oscillatory. The technique of the proof of our theorems depends mainly on the assumption that there exists an nonoscillatory solution of the equation (I), which may be assumed to be positive and then come out with a contradiction with some of our hypotheses. Previously, in 1982, Kwong and Wong [11] studied the oscillation of the equation (I) with $r(t) = 1$ and $0 < \gamma < 1$ and obtained some interesting criteria which extended by Philos [16].

In view of the present development it will be of interest to improve the above mentioned works by using an averaging criterion that introduced by Kamenev [11].

For additional results on the oscillatory behavior of solutions of such equations, we choose to refer the reader to the papers of [4], [6], [7], [17-19], and the references contained therein. See also the monograph of Lakshmikantham et. al. [13], and the recent paper of Ahmed et. al. [1] which discussed some open problems in the oscillation theory of functional nonlinear differential equations. The purpose of this study is to derive some new conditions under which all solutions of the equation (I) are oscillatory. In particular, the results obtained here are for the sublinear case, that is, for $0 < \gamma < 1$. In some sense, our results here extend and improve some of those available in the papers mentioned above.

Main Results

In this section we shall state and prove some sufficient oscillation criteria of the solution of the equation (I).

Theorem 2.1 Suppose that

(1) $0 < A \le r(t) \le B,\ t \ge t_0 \ge 0$,

(2) $\dot r(t) > 0,\ t \ge t_0 \ge 0$,

(3) $\lim_{t\to\infty} sup \int_{t_0}^{t} q(s)ds = \infty$

(4) $\lim_{t\to\infty} sup \int_{t_0}^{t}\int_{t_0}^{t} q(u)duds = \infty$

Then the equation (I) is oscillatory when $0 < \gamma < 1$.

Proof. Assume the contrary, then there exists a solution $x(t)$ which may be assumed to be positive on $[T_1, \infty)$ for some $T_1 \ge t_0 \ge 0$. We distinguish three cases for the behavior of $\dot x(t)$:

(i) $\dot x(t)$ is oscillatory on $[T_1, \infty)$.

(ii) $\dot x(t) > 0$ on $[T_2, \infty)$ for some $T_2 \ge T_1$.

(iii) $\dot x(t) < 0$ on $[T_2, \infty)$ for some $T_2 \ge T_1$.

Suppose that case (i) holds; then there exists a sequence $\{t\}_{n=1}^{\infty}$ such that $\dot x(t_n) = 0$ and $t \to \infty$ as $n \to \infty$. Dividing (I) through by $x^{\gamma}(t)$ and intergrating from t_k to t where k is some integer, we obtain

$$\frac{r(t)\dot x(t)}{x^{\gamma}(t)} + \gamma \int_{t_k}^{t} \frac{r(s)\dot x^2(s)}{x^{\gamma}} ds + \int_{t_k}^{t} q(s)ds = 0, \tag{2.1}$$

where $\frac{r(t_k)\dot x(t_k)}{x^{\gamma}(t_k)} = 0$. Now, from the condition (1) and the equation (2.1) we obtain that

$$\frac{r(t)\dot x(t)}{x^{\gamma}(t)} + \gamma A \int_{t_k}^{t} \left(\frac{\dot x(s)}{x^{\beta}(s)}\right)^2 ds + \int_{t_k}^{t} q(s)ds \le 0, \tag{2.2}$$

where $\beta = \frac{(\gamma+1)}{2}$. Integration the inequality (2.2) once more from t_k to t as follows:

$$\int_{t_k}^{t} r(s)\frac{\dot x(s)}{x^{\gamma}(s)} ds + \gamma A \int_{t_k}^{t}\int_{t_k}^{s} \left(\frac{\dot x(u)}{x^{\beta}(u)}\right)^2 duds + \int_{t_k}^{t}\int_{t_k}^{s} q(u)duds \le 0 \tag{2.3}$$

By condition (2), since $r(t)$ is non decreasing on$[t_0, \infty)$. So, by the Bonnet's theorem [3], for any $t \ge t_k (k\ is\ some\ integer)$, there exists $t_* \in [t_k, t]$ such that

$$\int_{t_k}^{t} r(s)\frac{\dot x(s)}{x^{\gamma}(s)} ds = r(t)\int_{t_*}^{t} \frac{\dot x(s)}{x^{\gamma}(s)} ds = \frac{r(t)}{1-\gamma}[x^{-\gamma+1}(t) - x^{-\gamma+1}(t_*)]$$

Substituting in the inequality (2.3), we obtain that

$$r(t)\frac{x^{-\gamma+1}(t)}{1-\gamma} + \gamma A \int_{t_k}^{t}\int_{t_k}^{s} \left(\frac{\dot x(u)}{x^{\beta}(u)}\right)^2 duds + \int_{t_k}^{t}\int_{t_k}^{s} q(u)\,duds \le r(t)\frac{x^{-\gamma+1}(t_*)}{1-\gamma} \tag{2.4}$$

By condition (1), since $0 < A \le r(t) \le B$, then, by substituting in the inequality (2.4) we obtain

$$\frac{A}{1-\gamma} x^{-\gamma+1}(t) + \gamma A \int_{t_k}^{t}\int_{t_k}^{s} \left(\frac{\dot x(u)}{x^{\beta}(u)}\right)^2 duds + \int_{t_k}^{t}\int_{t_k}^{s} q(u)\,duds \le C_1 \tag{2.5}$$

where $C_1 = \frac{B}{1-\gamma} x^{-\gamma+1}(t_*)$. Therefore (2.5) yields

$$\int_{t_k}^{t}\int_{t_k}^{s} q(u)\,duds \le C_1$$

Now, taking the upper limit as $t \to \infty$, we get

$$\lim_{t\to\infty} sub \int_{t_k}^{t}\int_{t_k}^{s} q(u)\,duds < \infty,$$

which contradicts the condition (4). Next, suppose that $\dot x(t) > 0\ for\ t \ge T_2 \ge T_1$.

Dividing the equation (I) through by $x^{\gamma}(t)$ and integrating from $T_2 \ge T_1$ to t to obtain

$$\frac{r(t)\dot x(t)}{x^{\gamma}(t)} + \gamma \int_{T_2}^{t} r(s)\frac{\dot x^2(s)}{x^{\gamma+1}(s)} ds + \int_{T_2}^{t} q(s)ds = C_2 , \tag{2.6}$$

where $C_2 = \frac{r(T_2)\dot x(T_2)}{x^{\gamma}(T_2)}$. From the condition (1) in the equation (2.6) we get

$$A\frac{\dot x(t)}{x^{\gamma}(t)} + \gamma A \int_{T_2}^{t} \frac{\dot x^2(s)}{x^{\gamma+1}(s)} ds + \int_{T_2}^{t} q(s)\,ds \le C_2. \tag{2.7}$$

Then, for all $t \ge T_2$ we have

$$\int_{T_2}^{t} q(s)ds \le C_2 \tag{2.8}$$

By taking the upper limit as $t \to \infty$, we get

$$\lim_{t\to\infty} sup \int_{T_2}^{t} q(s)ds < \infty,$$

which contradiction the condition (3). Finally, we assume that $\dot x(t) < 0\ for\ t \ge T_2 \ge T_1$.

Then from the condition (1), we have

$$\dot x(t)r(t) \ge B\dot x(t)$$

Thus, from the equation (2.6) we obtain

$$B\frac{\dot x(t)}{x^{\gamma}(t)} + \gamma A \int_{T_2}^{t} \left(\frac{\dot x(s)}{x^{\beta}(s)}\right)^2 ds + \int_{T_2}^{t} q(s)\,ds \le C_2 \tag{2.9}$$

Since $\gamma A \int_{T_2}^{t} \left(\frac{\dot x(s)}{x^{\beta}(s)}\right)^2 ds > 0$, then from the inequality (2.9) we obtain

$$B\frac{x^{1-\gamma}(s)}{1-\gamma}\Big|_{T_2}^{t} + \int_{T_2}^{t}\int_{T_2}^{s} q(u)\,duds \le C_2(t - T_2).$$

Then, for all $t \ge T_2$ we have

$$B\frac{x^{1-\gamma}(t)}{1-\gamma} - B\frac{x^{1-\gamma}(T_2)}{1-\gamma} + \int_{T_2}^{t}\int_{T_2}^{s} q(u)\,duds \le C_2(t - T_2),$$

which implies that

$$\int_{T_2}^{t}\int_{T_2}^{s} q(u)\,duds \le C_2(t - T_2) + B\frac{x^{1-\gamma}(T_2)}{1-\gamma}, \quad C_2 < 0$$

Taking the upper limit as $t \to \infty$, we obtain

$$\lim_{t\to\infty} sub \int_{T_2}^{t}\int_{T_2}^{s} q(u)\,duds = -\infty,$$

which again contradicts the condition (4). This completes the proof.

Example 2.1: Consider the equation

$$\left(\left(2 + \frac{e^t}{e^t+1}\right)\dot x(t)\right)^{\bullet} + (1 + 2\sin t)|x(t)|^{\gamma} signx(t) = 0, \quad t \ge t_0 \ge 0, 0 < \gamma < 1$$

Note that

(i) $2 < r(t) = 2 + \frac{e^t}{e^t+1} < 3,\ t \ge 0$

(ii) $\dot r(t) = \frac{e^t}{(e^t+1)^2} > 0, \quad t \ge 0$

(iii) $\lim_{t\to\infty} sup \int_{t_0}^{t} q(s)ds = \lim_{t\to\infty} sup \int_{t_0}^{t} [1 + 2\sin s]ds = \infty$

(iv) $\lim_{t\to\infty} sup \int_{t_0}^{t}\int_{t_0}^{t} q(u)\,duds = \lim_{t\to\infty} sub \int_{t_0}^{t} [s - 2\cos s - t_0 + 2\cos t_0]\,ds = \infty$

Hence, Theorem 2.1 ensures that the given equation is oscillatory for all $0 < \gamma < 1$.

Theorem 2.2: Suppose that conditions (1), (3) and (4) holds and that

(5) $\dot r(t) \le 0 \quad for \quad t \ge t_0$

Then the equation (I) is oscillatory for $0 < \gamma < 1$.

Proof. Assume the contrary, then there exists a solution $x(t)$ which may be assumed to be positive on $[T_1,\infty)$ for some $T_1 \geq t_0 \geq 0$. As in the proof of Theorem 2.1, (case 1), we obtain

$$\int_{t_k}^{t} r(s)\frac{\dot{x}(s)}{x^{\gamma}(s)}ds + \gamma A\int_{t_k}^{t}\int_{t_k}^{s}\left(\frac{\dot{x}(u)}{x^{\beta}(u)}\right)^2 duds + \int_{t_k}^{t}\int_{t_k}^{s} q(u)duds \leq 0$$

Now, by taking into account the condition (5), that is, $r(t)$ is non-increasing on$[t_0,\infty)$, we conclude, from the Bonnet's theorem [3], for any $t \geq t_k$ $(k \text{ is some integer})$, there exists $\alpha_t \in [t_k,t]$ such that

$$\int_{t_k}^{t} r(s)\frac{\dot{x}(s)}{x^{\gamma}(s)}ds = r(t_k)\int_{t_k}^{\alpha_t}\frac{\dot{x}(s)}{x^{\gamma}(s)}ds = \frac{r(t_k)}{1-\gamma}[x^{-\gamma+1}(\alpha_t) - x^{-\gamma+1}(t_k)]. \quad (2.10)$$

Using (2.10) in (2.3), we have

$$r(t_k)\frac{x^{-\gamma+1}(\alpha_t)}{1-\gamma} + \gamma A\int_{t_k}^{t}\int_{t_k}^{s}\left(\frac{\dot{x}(u)}{x^{\beta}(u)}\right)^2 duds + \int_{t_k}^{t}\int_{t_k}^{s} q(u)du\,ds \leq r(t_k)\frac{x^{-\gamma+1}(t_k)}{1-\gamma}. \quad (2.11)$$

Using the condition (1) in the inequality (2.4) we have

$$\frac{A}{1-\gamma}x^{-\gamma+1}(\alpha_t) + \gamma A\int_{t_k}^{t}\int_{t_k}^{s}\left(\frac{\dot{x}(u)}{x^{\beta}(u)}\right)^2 duds + \int_{t_k}^{t}\int_{t_k}^{s} q(u)duds \leq C_1', \quad (2.12)$$

where $C_1' = \frac{r(t_k)}{1-\gamma}x^{-\gamma+1}(t_k)$, and hence (2.12) yields

$$\int_{t_k}^{t}\int_{t_k}^{s} q(u)duds \leq C_1' \quad (2.13)$$

which contradicts the condition (4). The proofs of cases 2 and 3 are immediate consequences of cases 2 and 3 of Theorem 2.1 and so will be omitted. The proof is complete.

Example 2.2: Consider the following differential equation

$$\left[\left(8 + {e^{-t}}/{(e^{-t}+1)}\right)\dot{x}(t)\right]^{\bullet} + \left(\frac{1}{2} - 3\sin t\right)|x(t)|^{\gamma} sing(t) = 0, t \geq t_0 \geq 0,$$

where $0 < \gamma < 1$

Note that

(i) $8 < r(t) = 8 + \frac{e^{-t}}{e^{-t}+1} < 9$,

(ii) $\dot{r}(t) = \frac{-e^{-t}}{(e^{-t}+1)^2} < 0 <$ $for\ all$ $t \geq t_o \geq 0$,

(iii) $\lim_{t\to\infty} sup\int_{t_0}^{t} q(s)ds = \lim_{t\to\infty} sup\int_{t_0}^{t}({1}/{2} - 3\sin s)ds = \infty$

(iv) $\lim_{t\to\infty} sup\int_{t_0}^{t}\int_{t_0}^{t} q(u)duds = \lim_{t\to\infty} sup\int_{t_0}^{t}\int_{t_0}^{t}\left(\frac{1}{2} - 3\sin u\right)duds = \infty$

Hence, by Theorem 2.2, we conclude that the given equation.

Remark 2.1: Theorems 2.1 and 2.2 extend results of Wong [20], Onose [16] and Nasr [15].

Conclusion

In a conclusion, we established some oscillation theorems for a class of nonlinear differential equations. Several conditions for the oscillation of all solutions are obtained. These conditions extend and improve some of well known results in the literature. Some applications of the new results are listed.

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