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Some types of regular spaces cleavability (Splittability)

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Abstract In this paper we studied the cleavability over these topological spaces : D- regular space, D - completely regular space , D_σ-completely regular space and weakly regular space as following:
If $\mathcal{P}$ is a class of topological spaces with certain properties and if X is cleavable over $\mathcal{P}$ then $X \in \mathcal{P}.also$ If $\mathcal{P}$ is a
Class of topological spaces with certain properties and if Y is cleavable over $\mathcal{P}$ then $Y \in \mathcal{P}$
Keyword: D - regular space , D - completely regular space , D_σ- completely regular space , weakly regular space, absolutely cleavable.

انشقاق (انشطار) بعض انواع الفضاءات المنتظمة

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الملخص في هذا البحث درسنا حالة الانشقاق(او الانشطار) على بعض الفضاءات الطوبولوجية :
(D regular space, D - completely regular space, D_σ -completely regular space and weakly regular space):كالتالى
إذا كان $\mathcal{P}$ فصل من الفضاءات الطوبولوجية ذات خصائص معينة و كانت X قابلة للأنشطار على $\mathcal{P}$ فإن $X \in \mathcal{P}$. كذلك إذا كان $\mathcal{P}$ فصل من الفضاءات الطوبولوجية ذات خصائص معينة و كانت Y قابلة للأنشطار على $\mathcal{P}$ فإن $Y \in \mathcal{P}$
الكلمات المفتاحية D - regular space , D - completely regular space , D_σ- completely regular space , weakly regular space, absolutely cleavable.

Introduction

Different types of cleavability (originally named "splittability") of a topological space were introduced by Arhangl' Skii (1985) as following: A topological space X is said to be cleavable (or splittable) over a class of spaces $\mathcal{P}$ if for $A \subset X$ there exists a continuous mapping $f: X \longrightarrow Y \in \mathcal{P}$ such that $f^{-1} f(A) = A$,$f(X) = Y$. Throughout this paper X, Y will always denote the topological spaces on which no separation axioms are assumed, unless otherwise mentioned. Let A be a subset of X , $\mathrm{cl}\,A$ and $\mathrm{int}\,A$ denote the closure and interior of A , respectively. Definitions and some properties of some regular spaces as D - regular space, D - completely regular space, D_σ-completely regular space and weakly regular space are introduced in [5],[6] .

Preliminaries

Now we recall some definitions which we needed in this paper.

Definition (1)

A topological space X is said to be absolutely cleavable over a class of spaces $\mathcal{P}$ if $A \subset X$ and there exists an injective continuous mapping $f: X \longrightarrow Y \in \mathcal{P}$ such that $f^{-1} f(A) = A$.

Remark (1)

if $\mathcal{P}$ is the class of all spaces, we shall say that X is absolutely cleavable over $\mathcal{P}$. If f is an open ,closed ,perfect ,...(continuous) mapping , we shall say that X respectively open ,closed perfect absolutely cleavable over $\mathcal{P}$.

Note that if f is an injective continuous mapping of X into $Y \in \mathcal{P}$ then X is cleavable over $\mathcal{P}$ and since the definition of cleavability depends on the subset $A$of X , thus we might say a space X is said to be absolutely cleavable over $\mathcal{P}$, then the cleavability over $\mathcal{P}$ may regarded as generalization of continuous injection map onto $Y \in \mathcal{P}$.

Remark (2)

By an open [closed , perfect ,....] cleavable we mean that the continuous function $f: X \longrightarrow Y$ is an injective open [closed , perfect , ...] respectively

Definition (2)[5]

A topological space X is said to be D -regular if every point x of X has a neighborhood base consisting of open F_σ –sets

Definition (3)[5]

A topological space X is called weakly regular if every point x of X has a neighborhood base consisting of F_σ -sets.

Definition (4)

A collection $\mathfrak{B}$ of subsets of a space X is called an open Complementary system if $\mathfrak{B}$ consists of open sets of X such that for every $B \in \mathfrak{B}$ there exists $B_1, B_2, \dots \in B$ with $B = \cup\{X/B_i , i \in \mathrm{N}\}$

Definition (5)

A subset A of a space X is called a strongly open F_σ-set if there exists a countable open complementary system $\mathfrak{B}(A)$ with $A \in \mathfrak{B}(A)$).

the complement of strongly open F_σ -set is called strongly closed G_σ -set.

Definition (6) [5]

A topological space X is said to be D -completely regular if it has a base of strongly open F_σ - sets of X can be separated by G_σ -sets .

Proposition (1)

Let X be a closed absolutely cleavable space over a class of regular spaces , then $X \in \mathcal{P}$.

Proof:

Let x be any point in X , and F a closed subset of X ,with $x \notin F$,since X is absolutely cleavable , there exists an injective continuous mapping $f: X \longrightarrow Y \in \mathcal{P}$, such that $f^{-1}f(F) = F$, and for every $y \in Y$ there exists $x \in X$ such that $y = f(x) \Leftrightarrow f^{-1}(y) = x$.

Hence f(F) is closed subset of Y and $f(x) \notin f(F)$.

Now Y is regular, then there exist two open sets G and H of Y with $f(x) \in G$, $f(F) \subset H$, $G \cap H = \phi$ so $x \in f^{-1}(G)$, $f^{-1}f(F) \subset f^{-1}(H)$ this implies that $x \in f^{-1}(G)$, $F \subset f^{-1}(H)$,since f is continuous , then $f^{-1}(G)$, $f^{-1}(H)$ are open sets of X , and $f^{-1}(G) \cap f^{-1}(H) = f^{-1}(G \cap H) = f^{-1}(\phi) = \phi$ Therefore X is regular ,then $X \in \mathcal{P}$.

Proposition (2)

Let X be a regular space is an open absolutely cleavable over a class of spaces , then $Y \in \mathcal{P}$.

Proof:

Suppose y be any point in Y , and E be any closed subset of Y with $y \notin E$ then there exists $x \in X$, with $y = f(x) \Leftrightarrow f^{-1}(y) = x$,and an open Injective continuous mapping $f: X \longrightarrow Y \in \mathcal{P}$ such that $f^{-1}f(f^{-1}(E)) = f^{-1}(E)$.

Since f continuous then $f^{-1}(E)$ is closed in X this means that $f^{-1}(y) \notin f^{-1}(E)$, so $x \notin f^{-1}(E)$ in X .

Since X is regular space , then there exist open sets U ,V such that $x \in U$ and $f^{-1}(E) \subset V$ this implies that $f(x) = y \in f(U)$, and $ff^{-1}(E) \subset f(V)$ implies that $E \subset f(V)$, since f is open , so f(U) ,f(V) are open sets of Y and $f(U) \cap f(V) = f(U \cap V) = f(\phi) = \phi$.

Hence Y is regular space and $Y \in \mathcal{P}$.

Proposition (3)

Let X be D -regular closed open cleavable space over a class of Spaces $\mathcal{P}$, then $Y \in \mathcal{P}$.

Proof:

Suppose $y \in Y$ and a neighborhood V of Y $y = f(x) \Leftrightarrow f^{-1}(y) = x$ and an open closed continuous mapping $f: X \longrightarrow Y$ such that $f^{-1}f(f^{-1}(V)) = f^{-1}(V)$, then for

$x \in f^{-1}(V)$, $U = f^{-1}(V)$

is a neighborhood of x .

Since X is D -regular space , so there exists an open F_σ - set F such that $x \in F \subset U$,

since f is open then f(F) is an open F_σ - set in Y and $f(x) \in f(F) \subset f(U)$, this implies that $y \in f(F) \subset V$. therefore Y is D -regular space. Hence $Y \in \mathcal{P}$.

Proposition (4)

Let X be a Hausdorff perfect cleavable space over a class of weakly regular spaces $\mathcal{P}$, then $X \in \mathcal{P}$.

Proof:

Let U be an open neighborhood of $x \in X$, there exists a perfect mapping $f: X \longrightarrow Y$ such that $f^{-1}f(U) = U$, if $f^{-1}f(x) \subset U$

then exists an F_σ – set $\boldsymbol{F}$ in $\boldsymbol{Y}$ with

$f(x) \in \inf F \subset F \subset Y/f(X/U)$

which implies that $f^{-1}(F)$ is an F_σ - set , such that

$f^{-1}f(x) \in \operatorname{int} f^{-1}(F) \subset f^{-1}(F) \subset f^{-1}[Y/f(X/U)] \Rightarrow$

$x \in \operatorname{int} f^{-1}(F) \subset f^{-1}(F) \subset U$.

Hence $X \in \mathcal{P}$

Definition (7)

A mapping $f: X \longrightarrow Y$ is said to be Lindeloff –perfect if f is continuous , Closed and every set of the form $f^{-1}(y)$ for $y \in Y$ is Lindeloff spaces.

Lemma (1)[5]

Let A and B be two disjoint subsets of a space X with B is closed ,if A is a Lindeloff then there exists an F_σ - set F such that

$A \subset \operatorname{int} F \subset F \subset X/B$

proof:

Let A be closed and has a boundary

$K = A/\operatorname{int} A$ which is a Lindeloff space .then there exists countably many points $x_1, x_2, \ldots$ in K and F_σ-sets $F_1, F_2, \ldots$ such that $x_i \in \operatorname{int} F_i$ and $K \subset \operatorname{int} G \subset G = \cup\{F_i | _{i \in N}\} \subset X/B$

Therefore $F = A \cup G$ is an F_σ - set with

$A \subset \operatorname{int} F \subset F \subset X/B$

Proposition (5)

Let X be a weakly regular Lindeloff-perfect cleavable space over a class of spaces $\mathcal{P}$ .then $Y \in \mathcal{P}$

Proof:

Let V be an open neighborhood of $y \in Y$.then there exists a Lindeloff perfect mapping $f: X \longrightarrow Y$ such that $f^{-1}f\{f^{-1}(V)\} = f^{-1}(V)$, hence $f^{-1}(y)$ is a Lindeloff subspace of the open set $f^{-1}(V)$.

By lemma (1) there exists an F_σ - set F such that $f^{-1}(y) \subset \operatorname{int} F \subset F \subset f^{-1}(V)$

and since $U = Y/f[X/\operatorname{int} F]$ is open and

$f(f^{-1}(y)) \subset f(int\, F) \subset f(F) \subset V$,

then $y \in int\, f(F) \subset f(F) \subset V$.

Therefore Y is weakly regular.

hence $Y \in \mathcal{P}$.

Definition (8)

A subset A of a space X is called regular G_σ -set if A is an intersection of sequence of closed sets whose interiors contain A .

i-e $A = \cap_{n=1}^{\infty} F_n = \cap_{n=1}^{\infty} \operatorname{int} F_n$ where $\operatorname{int} F_n$ denotes the interior of F_n .

The complement of regular G_σ -set is called a regular F_σ - set

Definition (9) [5]

A topological space X is called a D_σ -completely regular space if it has a base of regula F_σ- sets

Proposition (6)[5]

Every D_σ -completely regular space is regular

Proof:

Let X be D_σ-completely regular space and let F be a closed subset of X and $x \notin F$ then $x \in X/F$and X/F is open set in X , so there exists regular F_σ-

set U such that $x \in U \subset X/F$, so U is regular F_σ-set then $U = \bigcup_{i=1}^{\infty} V_i = \bigcup_{i=1}^{\infty} cl\, V_i$ where each V_i is an open set . Hence $x \in V_i \subset cl\, V_i \subset X/F$which implies that V_i and $X/cl\, V_i$ are disjoint open sets containing x and F respectively.
Therefore X is regular space .
hence $X \in \mathcal{P}$

Proposition (7)

Let X be a D_σ -completely regular open closed cleavable space over a class of spaces .
then $Y \in \mathcal{P}$

Proof:

Let $y \in Y$ and V be a neighborhood of y, there exists an open closed continuous mapping $f: X \longrightarrow Y$ such that
$f^{-1}f\{f^{-1}(V)\} = f^{-1}(V)$, then for
$y = f(x) \Leftrightarrow x = f^{-1}(y)$, $U = f^{-1}(V)$ is a neighborhood of x . since X is D_σ- completely regular space , then there exists a regular F_σ- set F such that $x \in F \subset U$, since f is open closed then f(F) is a regular F_σ - set in Y with $f(x) \in f(F) \subset f(U)$
this implies that $y \in f(F) \subset f(U)$. Therefore Y is D_σ- completely regular space. Hence $Y \in \mathcal{P}$

conclusion

In this paper we have studied and proved these cases:

1) If $\mathcal{P}$ is a class of egular space with certain properties and if X is absolutely cleavable over $\mathcal{P}$, then $X \in \mathcal{P}$, also if $\mathcal{P}$ is class of regular space with certain properties and if Y is absolutely cleavable over $\mathcal{P}$, then $Y \in \mathcal{P}$.
2) If $\mathcal{P}$ is a class of weakly regular space or with certain properties and if X is cleavable over $\mathcal{P}$, then $X \in \mathcal{P}$.
3) If $\mathcal{P}$ is a class of $D-$ regular space or (weakly regular space , $D_\sigma -$ completely regular space) with certain properties and if Y is absolutely cleavable over $\mathcal{P}$, then $Y \in \mathcal{P}$.

References

[1]- Arhangel'skii,A.V and Cammaroto,F ,On different types of cleavability of topological spaces , pointwise, closed ,open and pseudo open , Journal of Australian Math,Soc(1992).

[2]- Arhangel'skii,A.V and Cammaroto,F., On cleavability and hereditary properties Houston journal of Mathematics ,20,(1994) .

[3]- Cammaroto, F and Lj, Kocinac, Develoable spaces and cleavability Rediconti di Mathemathica, serie VII ,vol,15 ,Roma (1995), 647-663.

[4]- Cammaroto.F., Cleavability and divisibility over developale spaces. Comment. Math.Univ,Carolinae 37,4 (1996) 791-796.

[5]- J.K.Kohli and D.Singh. Between regularity and complete regularity and a factorization of complete regularity . Seria: MATEMATICĂ Nr. 17(2007), pag. 125-134

[6]- N.,Helderman:Developablity and some new regularity a xioms, Can .j,Math ,33(1981) ,641-883.