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The Convergence rates of chebyshev Interpolation

Souad A. Abumaryam
Mathematics department, School of Science, Sirte University, Libya
Corresponding Author: souad22008@yahoo.com

Abstract In this paper we will investigate the convergence of Chebyshev interpolation in terms of Chebyshev polynomials. In particular, if the function $f(x)$ extends to an analytic function in a region bounded by an ellipse, then we may obtain an upper bound on the error of interpolation using zeros and extrema of Chebyshev polynomials.

Keywords: Chebyshev polynomial, Chebyshev interpolation, Convergence rate.

معدل التقارب لكثيرات حدود تشيبشيف الأستكمالية

سعاد أحمد أبومريم
قسم الرياضيات – كلية العلوم – جامعة سرت، ليبيا
للمراسلة: souad22008@yahoo.com

الملخص في هذه الورقة نتحقق من التقارب في حدودية تشيبشيف الأستكمالية بالتحديد اذا كانت الدالة تحليلية محدودة بقطع ناقص. ومن ثم نجد الحد الأعلى لدالة الخطأ باستخدام جدور كثيرة حدود تشيبشيف.

الكلمات المفتاحية: معدل التقارب، حدود تشيبشيف، حدودية تشيبشيف الأستكمالية.

Introduction

As known, the Chebyshev polynomial of the first kind of degree n is defined as:

$T_n(x) = \cos(n \cos^{-1} x) = \cos n\theta$, (0.1)

where $x = \cos\theta$, $-1 \le x \le 1$, $0 \le \theta \le \pi$, and n is a non negative integer [2].

The Chebyshev polynomials $T_n(x)$ satisfy $|T_n(x)| \le 1$.

This follows from the bound

$-1 \le \cos x \le 1$,

which leads to

$|T_{n+1}(x) - T_{n-1}(x)| \le 2$. (0.2)

The Chebyshev polynomial $T_n(x)$ of degree $n \ge 1$ has n zeros on the interval $[-1, 1]$. The zeros x_j are given by:

$x_j = \cos\left(\frac{(2j-1)}{2n}\right)\pi$, j=1,…n

Moreover, the extrema, or points x_j such that $T_n(x_j) = (-1)^j$ are given by:

$x_j = \cos\left(\frac{j\pi}{n}\right)$, j=1,…n

All roots are real and lie in the interval $[-1, 1]$.

The extrema are preferable for interpolation in practical uses, because they include the boundary points.

Theorem 0.1 *[4] A function $f(x)$ on $[-1,1]$ that satisfies the Lipchitz continuity condition can be expanded as a Chebyshev series*

$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k T_k(x)$, (0.3)

which converges uniformly and absolutely on $[-1,1]$, where

$a_k = \int_{-1}^{1} \frac{f(x)T_k(x)}{\sqrt{1-x^2}} dx, k \ge 1$ (0.4)

The Chebyshev polynomials have interesting properties that make them a very attractive tool to minimize the maximum error in uniform approximation.

1.Convergence Rate

The convergence of Chebyshev series is determined by a property of the function $f(x)$.

In [4], Trefethen has shown that if the function f is smooth, then its Chebyshev expansion coefficients decrease rapidly. Two notions of smoothness were considered: an r^{th} derivative with bounded variation, or analyticity in a neighborhood of $[-1,1]$.

Theorem1.1 [8,p.66] The truncation error when approximating a function $f(x)$ in terms of Chebyshev polynomials satisfies

$$|f(x) - f_n(x)| \le \sum_{k=n+1}^{\infty} a_k$$

If all a_k are rapidly decreasing, then the error is dominated by the leading term $a_{k+1}T_{k+1}$.

The coefficients a_k for $k > n+1$ are negligibly small, where the rest of the terms will be neglected if $a_{n+1} \ne 0$.

Theorem 1.2 [4, p.51] If $f, f', \dots, f^{(r-1)}$ are absolutely continuous for $r \ge 0$ on $[-1,1]$, where the r^{th} derivative $f^{(r)}$ has bounded variation $V = \|f^{(r)}\|$, then the coefficients of the Chebyshev series satisfy the fallowing inequality

$$|a_k| \le \frac{2V}{\pi n k(k-1)\dots(k-r)} \quad (1.5)$$

$for each k \ge r+1$.

Theorem 1.3 *[4, p.51] Let a function f be analytic on $[-1, 1]$ and analytically continuable to the ellipse $E_\rho := \{z \in C : z = \rho\,(e^{i\theta} + e^{-i\theta})/2 , \theta \in [0, 2\pi]\}$ in which $|f(z)| \le M$ for some M . For all $k \ge 0$ the Chebyshev coefficients a_k of f exponentially decay as $k \to \infty$ and satisfying*

$|a_k| \le 2M\rho^{-k}$, $\rho > 1$. (1.6)

Theorem 1.4 *[4, p.53] If f is absolutely continuous for $r \ge 0$ on $[-1, 1]$, where the r^{th} derivative $f(r)$ has bounded variation $V = \|f^{(r)}\|$, then the Chebyshev truncation satisfies*

$\|f - f_n\| \le \frac{2V}{\pi r(n-r)^r}$ (1.7)

Theorem 1.5 [4, p.58] Let a function f be analytic on *[−1, 1]* and analyticall continuable to the open ellipse E_ρ, in which $|f| \le M$ for some M . Then the Chebyshev truncation error satisfies

$$\|f - f_n\| \le \frac{2M\rho^{-n}}{\rho-1} \qquad (1.8)$$

2.Polynomial and Chebyshev interpolation

An interpolating polynomial can be constructed easily using the Lagrange formula [1]

$$p(x) = \sum_{j=0}^{n} f_j L_j(x) \qquad (2.9)$$

where the Lagrange polynomials basis $\boldsymbol{L_j}$ is:

$$L_j(x)=\prod_{\substack{k=0\\k\neq j}}^{n} \frac{x-x_k}{x_j-x_k} \qquad (2.10)$$

$\boldsymbol{L_j}$are polynomials of degree n that have the property

$$L_j(x_k) = \begin{cases} 1, if j = k, \\ 0, if j \neq k. \end{cases}$$

The Lagrange interpolation formula is useful for theoretical interest but not appropriate in practice [3]. Let us define

$$L(x)=\prod_{k=0}^{n}(x - x_k) \qquad (2.11)$$

If the interpolation at Chebyshev points or extrema ,then

$\boldsymbol{L(x)}=\boldsymbol{T_n}$or $\boldsymbol{L(x)} = \boldsymbol{T_{n+1}(x)} - \boldsymbol{T_{n-1}}(x)$

respectively.See also [4]

Given a function f that is interpolated at $n + 1$ points in term of Chebyshev polynomials and that satisfies the interpolation condition $p_n(x_j)=f(x_j)$, we have the following theorem:

Theorem 2.1 *[4] Let f (x) be a Lipschitz continuous function on* [−1, 1], *where*

$$f(x) = \sum_{k=0}^{\infty} a_k T_k(x), \qquad (2.12)$$

Then the function f *(x)* can be presented by interpolation in Chebyshev points as

$$p_n = \sum_{k=0}^{\infty\prime\prime} b_k T_k(x), \qquad (2.13)$$

where $b_k = \frac{2}{n}\sum_{j=0}^{n} f(x_j)T_k(x_j), x_j = cos\left(\frac{j\pi}{n}\right)$.

and

$$p_n = \sum_{k=0}^{\infty\prime\prime} c_k T_k(x), \qquad (2.14)$$

where $c_k = \frac{2}{n+1}\sum_{j=0}^{n} f(x_j)T_k(x_j), x_j = cos\left(\frac{(2j-1)}{2n}\right)\frac{n+1}{\pi}.$

Here a_k are the exact coefficients, and b_k and c_k are coefficients of p_n. The symbol ″ means that the first and last terms are halved.

The coefficients of truncated Chebyshev series and Chebyshev interpolations can be obtained by means of orthogonality properties of Chebyshev polynomials for both continuous and discrete. Now, we discuss the relations between the coefficients a_k , b_k and c_k.

Theorem*2.2* Let f(x) be a Lipschitz continuous function on *[−1, 1]* and let b_k and c_k be the coefficients of the Chebyshev interpolant defined in (2.13) and (2.14). Then and

$$b_k = a_k + \sum_{m=1}^{\infty} a_{2mn-k} + a_{2mn+k} \qquad (2.15)$$

$$c_k = a_k + \sum_{m=1}^{\infty} a_{2m(n+1)-k} + a_{2m(n+1)+k}\,(-1)^m$$

(2.16)

proof *By using (2.12) and (2.13) we have*

$$b_k = \frac{2}{n}\sum_{j=0}^{n\prime\prime} f(x_j)T_k = \frac{2}{n}\sum_{p=0}^{\infty} a_k \sum_{k=0}^{n\prime\prime} cos\frac{jk\pi}{n} cos\frac{jp\pi}{n},$$

By the orthogonality relation

$$\sum_{j=0}^{n} cos\frac{jk\pi}{n} cos\frac{jp\pi}{n} = \begin{cases} \frac{n}{2}, if p = 2mn \pm k, m = 1,2, \\ 0, otherwise \end{cases}$$

We have

$T_k(x_j) = cos\frac{(kj)\pi}{n}$=cos=

$= T_{2mn\pm k}(x_j).$

Therefore, we have the following relation between the coefficients a_k and b_k

$$b_k = a_k + \sum_{m=1}^{\infty} a_{2mn-k} + a_{2mn+k}$$

Since the infinite sequence b_k is semi-periodic with period 2n and $b_k=b_{2n-k}$, only the first n+*1* terms $b_0, b_1, \cdots, b_n$ *are unique.*

In case of Chebyshev zeros of the first kind, by the orthogonality relation

$$\sum_{j=0}^{n} cos\frac{jk\pi}{n} cos\frac{jp\pi}{n} =$$

$$\begin{cases} \frac{(-1)^m n + 1}{2}, if p = 2mn \pm k, m = 1,2, \dots \\ 0, otherwise \end{cases}$$

And we have

$T_k(x_j) = cos\frac{(2j-1)k\pi}{2n}$=

$= (-1)^m cos$

$= (-1)^m T_{2m(n+1)\pm k}(x_j)$

This comes from the fact that Chebyshev polynomials have the same values on n + *1* points

(T_k , T_{2n-k} ,T_{2n+k} , T_{4n-k} ,T_{4n+k} , ⋯).

Theorem 2.4 *[5] Assume that* x_j, j=0,…,n are distinct points in [a,b] and that f(x) is a function in C^{n+1} [a,b] and $|f^{n+1}| \le$ M. Let $p_n$be a sequence of polynomial interpolating f. Then for each x ϵ [a, b], there is $\zeta\epsilon$ (a, b) such that

$|f(x) - p_n(x)| \le$ (2.17)

The choice of Chebyshev points minimizes the terms $\prod_{k=0}^{n}(x - x_k)$ on [-1,1].

This choice ensures uniform convergence for a Lipschitz continuous function *f*. This condition is more important than the condition of continuity of the function *f*.

Theorem 2.5 *Let f (x) be a continuous function on [a, b] and let $p_n(x)$ be interpolant polynomials of f* at Chebyshev zeros. Then the error is given by

$$\|f - p_n\|_\infty \le \left\|\frac{2(b-a)^{n+1}}{4^{n+1}(n+1)!}\right\|_\infty \|f^{n+1}(\zeta)\|_\infty \qquad (2.18)$$

Similarly, the error at Chebyshev extrema is given by:

$$\|f - p_n\|_\infty \le \left\|\frac{1}{2^{n-1}(n+1)!}\right\|_\infty \|f^{n+1}(\zeta)\|_\infty \qquad (2.19)$$

Main result

Now, we will investigate the interpolation convergence bound at zeros and extrema of Chebyshev polynomials:

Theorem 2.6 Let f be a bounded analytic function such that |f *(z)*| ≤ M in the region bounded by an ellipse with foci ±*1* and major semi-axis a=$\frac{\rho+\rho^{-1}}{2}$ and minor semi-axis b= $\frac{\rho-\rho^{-1}}{2}$ summing to ρ >*1*. Then

$$\|f-p_n\|_\infty \le \frac{2M\sqrt{\rho^2+\rho^{-2}}}{\left(\frac{1}{2}(\rho+\rho^{-1})-1\right)(\rho^n-\rho^{-n})}$$

where $p_n(x)$ is the polynomial interpolant at Chebyshev zeros.

proof Let E_ρ be an ellipse enclosing the interval [−1, 1] in its interior and let f (z) be an analytic function within E_ρ. Since the points are zeros of Chebyshev polynomials, then the error of the Lagrange interpolation for f (x) at these points can be expressed as

$$f-p_n=\frac{T_n(x)}{2\pi i}\int_{E_\rho}\frac{f(z)}{T_n(z)(z-x)}dz \qquad (2.20)$$

Let $l(E\rho)$ *be the arc length of* $E\rho$ *and* $M=\max_{z\in E_\rho}|f(x)|$. Then $l(E\rho)$ is given by

$$l(E\rho)=\int_0^{2\pi}|z'(\theta)|d\theta$$

since

$$|z'(\theta)|=\left|\frac{i}{2}\left(\rho e^{i\theta}-\rho e^{-i\theta}\right)\right|\le\frac{1}{2}(\rho-\rho^{-1})$$

We obtain the estimation;

$$l(E\rho)=\pi\sqrt{\rho^2+\rho^{-2}}$$

For $z\epsilon E\rho, |z-x|$ is greater than or equal to the minimum distance from $E\rho$ to the interval *[−1, 1]*. Thus for $z\epsilon E\rho$

$$|z-x|\ge a-1=\frac{1}{2}(\rho+\rho^{-1})-1= 2\left(\frac{\rho^{1/2}-\rho^{-1/2}}{2}\right)^2.$$

To estimate $|T_n(z)|$, let $z=\frac{1}{2}\left(\rho^{i\theta}+\rho^{-i\theta}\right)$ Then we have

$T_n(z)=\frac{1}{2}\left(\rho^n e^{in\theta}+\rho^{-n}e^{-in\theta}\right)$.

Thus

$$T_n(z)=\frac{1}{2}[(\rho^n+\rho^{-n})cosn\theta+i(\rho^n-\rho^{-n})sinn\theta]$$

Then

$$|T_n(z)|=\frac{1}{2}\sqrt{\rho^{2n}+\rho^{-2n}+2cos2n\theta}$$

For $cos\theta=1$, we have $|T_n(x)|=\frac{1}{2}(\rho^n+\rho^{-n})$, and for $cos\theta=-1$, we have $|T_n(x)|=\frac{1}{2}(\rho^n-\rho^{-n})$. Then $\frac{1}{2}(\rho^n-\rho^{-n})\le|T_n(x)|\le\frac{1}{2}(\rho^n+\rho^{-n})$.

Therefore

$$\|f-p_n\|_\infty \le \frac{2M\sqrt{\rho^2+\rho^{-2}}}{\left(\frac{1}{2}(\rho+\rho^{-1})-1\right)(\rho^n-\rho^{-n})}$$

We can also obtain a slightly different estimation:

Theorem 2.8 Let f be a bounded analytic function such that |f (z)| ≤ M in the region bounded by an ellipse with foci ±1 and major and minor semi-axes summing to ρ >1. Then

$$\|f-p_n\|_\infty \le \frac{2M\sqrt{\rho^2+\rho^{-2}}}{\left(\frac{1}{2}(\rho+\rho^{-1})-1\right)(\rho-\rho^{-1})(\rho^n-\rho^{-n})}$$

where $p_n(x)$ is the polynomial interpolant at Chebyshev extrema.

proof In order to obtain the estimation we choose

$\emptyset_n(x)=T_{n+1}(x)-T_{n-1}(x)$

The error formula for extrema is

$$f-p_n=\frac{T_{n+1}(x)-T_{n-1}(x)}{2\pi i}\int_{E_\rho}\frac{f(z)}{[T|n+1(x)-T_{n-1}(x)](z-x)}dz$$

Since $\|T_n(x)\|_\infty$=1, we have that $\|\emptyset_n(x)\|_\infty\le 2$ and thus

$$|f-p_n|\le\left|\frac{2}{\pi}\int_{E_\rho}\frac{f(z)}{\emptyset_n(z)(z-x)}dz\right|$$

Now, in order to estimate $T_{n+1}(x)-T_{n-1}(x)$, we observe that

$$T_{n+1}(x)-T_{n-1}(x)=\frac{1}{2}[(\rho-\rho^{-1})cos\theta+i(\rho+\rho^{-1})sin\theta][(\rho^n-\rho^{-n})cosn\theta+i(\rho^n+\rho^{-n})sinn\theta]$$

Taking the absolute value, we obtain

$$|T_{n+1}(x)-T_{n-1}(x)|=\sqrt{\rho^2+\rho^{-2}-2cos2n\theta}\sqrt{\rho^{2n}+\rho^{-2n}-2cos2n\theta}$$

Therefore, the lower bound is achieved when $cos\theta=1$, and so

$$|\emptyset_n(x)|=\frac{1}{2}(\rho-\rho^{-1})(\rho^n-\rho^{-n})$$

and the upper bound is achieved when $cos\theta=-1$, and so

$$|\emptyset_n(x)|=\frac{1}{2}(\rho+\rho^{-1})(\rho^n+\rho^{-n}).$$

Then

$$\|f-p_n\|_\infty\le\frac{2M\sqrt{\rho^2+\rho^{-2}}}{\left(\frac{1}{2}(\rho+\rho^{-1})-1\right)(\rho-\rho^{-1})(\rho^n-\rho^{-n})}, \rho>1.$$

Finally we provide a simpler bound.

Theorem 2.10 Let f (x) be an absolutely continuous function on [−1, 1], and let p_n interpolates the function f (x) in term of Chebyshev polynomials. Then

$\|f-p_n\|\le\sum_{k=n+1}^{\infty}a_k=\frac{4V}{\pi kn(n-1)\ldots\ldots(n-r+1)}$

Moreover, if the function f is an analytic for function which |f(z)|≤M in domain bounded by the ellipse with foci ± 1 and major and minor semi-axis add to ρ >1, then for n ≥0,

$$\|f-p_n\|\le\frac{4M}{(\rho-1)\rho^{-n}}$$

proof *We start as*

$f-p_n=\sum_{k=0}^{n-1}-b_kT_k(x)+\left(a_n-\frac{b_n}{2}\right)T_n(x)+\sum_{k=n+1}^{\infty}a_kT_k(x)$

where a_k , b_k and c_k are defined in (0.4), (2.15) and (2.16).

There are three error terms in the above expression, the last term of which is a truncated error term.

Therefore, since $\|T_n(x)\|_\infty$=1, we have

$$\|f-p_n\|\le\sum_{k=0}^{n-1}|a_k-b_k|\,\|T_k\|_\infty+\left|a_n-\frac{b_n}{2}\right|\|T_n\|_\infty+\sum_{k=n+1}^{\infty}|a_k|\,\|T_k\|_\infty$$

$$\sum_{k=0}^{n-1}|a_k-b_k|+\left|a_n-\frac{b_n}{2}\right|+\sum_{k=n+1}^{\infty}|a_k|$$

The relation (2.15) yields

$\sum_{k=0}^{n-1}|a_k-b_k|+\left|a_n-\frac{b_n}{2}\right|\le\sum_{k=n+1}^{\infty}|a_k|$

Then, combining this with the above, we have

$$\sum_{k=0}^{n-1}|a_k-b_k|+\left|a_n-\frac{b_n}{2}\right|+\sum_{k=n+1}^{\infty}|a_k|\le 2\sum_{k=n+1}^{\infty}|a_k|$$

For Chebyshev zeros, by using (2.16)

$$\|f-p_n\| \leq \sum_{k=0}^{n-1} |a_k - c_k| \, \|T_k\|_\infty + \sum_{k=n+1}^{\infty} |a_k| \, \|T_k\|_\infty \leq$$

$$\leq 2 \sum_{k=n+1}^{\infty} |a_k|$$

$$\leq \sum_{k=0}^{n} |a_k - c_k| + \sum_{k=n+1}^{\infty} |a_k| \leq 2 \sum_{k=n+1}^{\infty} |a_k|$$

From this and by the Theorem 1.2, we get

$$\|f-p_n\| \leq \sum_{k=n+1}^{\infty} |a_k| \leq \sum_{k=n+1}^{\infty} \frac{2V}{\pi k(k-1)\ldots(k-r)} =$$

$$\frac{2V}{\pi n(n-1)\ldots(n-r)}$$

And for an analytic function, by the Theorem 1.3

$$\|f-p_n\| \leq \sum_{k=n+1}^{\infty} |a_k| \leq \sum_{k=n+1}^{\infty} \frac{2M}{\rho^K} = \frac{2M}{(\rho-1)\rho^n}.$$

Conclusion: We investigated polynomial convergence rates for different classes of functions. The result that obtained show that: For an r times differentiable function f with the r^{th} derivatives of bounded variation, the polynomial interpolant p_n with Chebyshev points in [-1,1] converges at an algebraic rate. For a function which is holomorphic and bounded in certain ellipse, the polynomial interpolant at Chebyshev points converges exponentially in the uniform norm.

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