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New Types of β- Generalized and β- Separate Axioms for Topological Spaces

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Abstract The aim of this paper is to present and study some new types of separation axioms called $\boldsymbol{\beta}$-generalized separation axioms, termed by generalized $\boldsymbol{\beta}$-$\boldsymbol{R_i}, \boldsymbol{i = 0, 1}$ and generalized $\boldsymbol{\beta}$-$\boldsymbol{T_i}, \boldsymbol{i = 0, 1, 2}$. The axioms were presented by using generalized $\boldsymbol{\beta}$- open sets in topological spaces according to Ganster and Steiner. The study Showed the connections, properties and characteristics between these axioms provided with examples and theorems.

Key Words: β- open sets, β- closed sets, $g\beta$- sets, β- ***continuous*** and β-T_k, $(k = 0, 1, 2)$.

أنواع جديدة من β المعممة والبديهيات المنفصلة β للفضاءات التوبولوجية

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الملخص الهدف من هذه الورقة هو عرض ودراسة بعض الأنواع الجديدة من بديهيات الفصل التي تسمى بديهيات الفصل المعممة ، والتي يطلق عليها generalized $\boldsymbol{\beta}$-$\boldsymbol{R_i}, \boldsymbol{i = 0, 1}$ و generalized $\boldsymbol{\beta}$-$\boldsymbol{T_i}, \boldsymbol{i = 0, 1, 2}$. تم تقديم البديهيات بإستخدام مجموعات عامة مفتوحة في الفضاءات التوبولوجية وفقاً لـــ Ganster و Steiner. بينت الدراسة الروابط والخصائص بين هذه البديهيات المقدمة مع الأمثلة والنظريات.

الكلمات المفتاحية: المجموعات المفتوحة β ، المجموعات المغلقة β ، مجموعات $g\beta$ ، إستمرارية β ، $\beta - T_k, (k = 0, 1, 2)$.

1- Introduction

In the mathematical paper [18] introduced and defined an ∝- open closed set. After the works of Njastad O. on ∝ - open sets, various mathematicians turned their attention to the generalizations of various concepts in topology by considering semi-open, ∝- open sets. The concept of ∝- closed [11], s- open [10] and ∝- open [18] sets have a significant role in the generalization of continuity in topological spaces. In [19] introduced semi-open sets in topological spaces and these semi-open sets were used to define three new separation axioms called semi -T_0, semi -T_1 and semi -T_2 by [13]. Abd El-Monsef et al. [2] introduced the notion of β- open sets and β-continuity in topological spaces. The concepts of Z*- open set and Z*- continuity introduced by Mubarki [15]. In 1996, Andrijevic [6] introduced a class of open sets called b- open sets in topology. In1970 Levine [11] introduced the concept of generalized closed sets. Ganster and Steiner [8] generalized the concept of closed sets to b-generalized closed sets and generalized b- closed sets. The investigation on generalization of closed set has lead to significant contribution to the theory of separation axioms. Navalagi [16] introduce semi generalized - T_i spaces, i = 0, 1, 2. This paper is devoted to introduce a new class of separation axioms called generalized β-R_i (briefly $g\ \beta$-R_i), i = 0 , 1 and generalized β-T_i (briefly $g\beta$-T_i), i = 0, 1, 2 axioms using $g\beta$- open sets due to Ganster and Steiner [8]. We study basic properties and preservation properties of these spaces. After that there is a vast progress occurred in the field of generalized open sets (compliment of respective closed sets) which became the base for separation axioms in the respective context.

Definition 1.1[1]

A subset A of topological space (X, τ) is called β- open or (semi pre open[5]) set if $A \subset cl\big(int(cl(A))\big)$. The complement of β-open sets is called β- closed sets.

Definition 1.2 [3]

For subset A in a topological space (X, τ), then the β- ***closure*** of A is the intersection of all β-***closed*** sets that contain A, and denoted by $\beta cl(A)$.

Definition 1.3 [3]

The β- ***interior*** of A is the union of all β-open sets that contained in A, and denoted by $\beta int(A)$.

Theorem 1.1 [4]

If A is a subset of a topological space (X, τ), then

1- A is β- open iff $A \cap \beta b(A) = \emptyset$.

2- A is β- closed iff $\beta b(A) \subseteq A$.

3- A is β- open and β-closed iff $\beta b(A) = \emptyset$.

Definition 1.4 [14]

A topological space (X, τ) is said to be β- T_0 if to each pair of distinct points x, y of X there exists a β- open set A containing x but not y or a β- open (resp. ***semi***- open) set B containing y but not x.

Definition 1.5 [14]

A topological space (X, τ) is said to be β- T_1 if to each pair of distinct points x, y of X, there exists a pair of β- open sets, one containing x but not y and the other containing y but not x.

Definition 1.6 [14]

A topological space (X,τ) is said to be β- T_2 if to each pair of distinct points x,y of X, there exists a pair of disjoint β- open sets, one containing x and the other containing y.

Definition 1.7 **[17]**

A function $f: X \longrightarrow Y$ is called β- ***continuous*** if the inverse image of each open subset of Y is β- open in X.

Theorem 1.2 **[17]**

If $f: X \longrightarrow Y$ is β- ***continuous*** function and X_0 is an open set in X, then the restriction $f/X_0 : X_0 \longrightarrow Y$ is β- ***continuous*** function.

Theorem 1.3

A topological space X is β-T_0 if and only if $\beta cl\{x\} \neq \beta cl\{y\}$ for every pair points $x,y \in X, x \neq y$.

Proof

$\Rightarrow$ Let $x \,\&\, y \in \beta$ - T_0 space, $x \neq y$, (By hypothesis) suppose that $U \in \beta O(X)$ such that $x \in U, y \notin U$. Hence $y \in X - U \,\&\, X - U$ is β - ***closed*** set. There for $\beta cl\{x\} \neq \beta cl\{y\}$.

$\Leftarrow$ Suppose that $z \in \beta cl\{x\}$ but $z \notin \beta cl\{y\}$, if $x \in \beta cl\{y\}$ then $\{x\} \subset \beta cl\{y\}$ which implies that $\beta cl\{x\} \subset \beta cl\{y\}$.This is a contradiction, since $z \in \beta cl\{x\}$. Then $x \in \beta$-***open*** set $\left[\beta cl\{y\}\right]^C$ which is $y \notin \left[\beta cl\{y\}\right]^C$, then the space is β-T_0 space.

Theorem 1.4

A space (X,τ) is β-T_1 if and only if $\{x\}$ is β- closed for all $x \in X$.

Proof

$\Rightarrow$ Suppose that X is β-T_1 space. If $y \notin \{x\}$ then $y \neq x$, since X is β-T_1 space, so there exist an β-***open*** set V such that $y \in V$ and $x \notin V$, so V is an β-***open*** set containing y and $V \cap \{x\} = \emptyset$, so $y \in V \subset \{x\}^C$ i.e. $\{x\}^C = \cup\{V: y \in \{x\}^C\}$ i.e. $\{x\}^C$ is an β- ***open***, then $\{x\}$ is a β- closed set.

$\Leftarrow$ Suppose that $\{x\}$ is β- closed set for all $x \in X$, let $x,y \in X$ with $x \neq y$, then $y \in \{x\}^C$ Hence $\{x\}^C$ is β- ***open*** set containing y but not x, also $\{y\}^C$ is β-***open*** set containing x but not y.Then X is an β-T_1 space.

Theorem 1.5

A space (X,τ) is β-T_1 if and only if $A = \cap\{U: U \text{ is } \beta - open, A \subset U\}$ for any $A \subset X$.

Proof

$\Rightarrow$ Suppose that X is β-T_1 space, and let $A \subset X$, $\{x\}$ is closed, clearly that $A \subset \cap\{U: U \text{ is } \beta - open, A \subset U\}$, since X is β - T_1, $x, y \in X \ni x \neq y, x \notin A$, and $\{x\}$ is closed, then $\{x\}^C$ is open set and since $x \notin A$, so $A \subset \{x\}^C$, there for $\{x\}^C$ is an open set containing A and $x \notin \{x\}^C$ so $x \notin \cap\{U: U \text{ is } \beta - open, A \subset U\}$, so $\cap\{U: U \text{ is } \beta - open, A \subset U\} \subset A$, and hence $A = \cap\{U: U \text{ is } \beta - open, A \subset U\}$.

$\Leftarrow$ Let $x \in X, x \neq y$, since $A = \cap\{U: U \text{ is } \beta - open, A \subset U\}$ then $\{x\} = \cap\{U: U \text{ is } \beta - open, \{x\} \subset U\}$, $\{y\} = \cap\{V: V \text{ is } \beta - open, \{y\} \subset V\}$ there for there exist an β- ***open*** set U containing x and not y, also there exist an β- ***open*** set V containing y and not x, so X is an β-T_1 space.

Theorem 1.6

Every discrete space is β-T_2 space.

Corollary 1.1

Notice that

1- β-$T_2 \Rightarrow \beta$-$T_1 \Rightarrow \beta$-T_0, but β-$T_0 \nRightarrow \beta$-$T_1 \nRightarrow \beta$-T_2.

2- Every sub space of β-T_i space is a β-T_i space, $i = 0, 1, 2$.

Theorem 1.7

Every subspace of β - T_2 space is a β - T_2 space.

Proof

Suppose that U be an open sub space of a β-T_2 space (X,τ), let $x \,\&\, y$ be any two distinct points of U, since X is β-T_2 and $U \subseteq X$, then there exist two β- open sets G and W such that $G \neq W$ and $x \in G, y \in W, G \cap W = \emptyset$. Let $A = U \cap G$ and $B = U \cap W$, then $A \text{ and } B$ are β - open sets in U containing $x \text{ and } y, A \cap B = \emptyset$ .There fore (U,τ_0) is β-T_2 space.

Theorem 1.8

If X_j is a β-T_i space for all $j \in \Gamma$, then $\prod_{j\in\Gamma} X_j$ is a β-$T_i$space, for $i = 0, 1, 2$.

Proof

Let X_j be β-T_2 a space for all $j \in \Gamma$ and let $x, y \in \prod_{j\in\Gamma} X_j, x \neq y$, since $x \neq y$ so there exist $\gamma \in \Gamma$ such that $x_\gamma \neq y_\gamma$. Since X_γ is a β-T_2 space, $x_\gamma \neq y_\gamma$ in X_γ, so there are two β- open sets U, V in X_γ such that $x_\gamma \in U, y_\gamma \in V$ and $U \cap V = \emptyset$, then $\prod_\gamma^{-1}(U), \prod_\gamma^{-1}(V)$ are β- open in $\prod_{j\in\Gamma} X_j, x \in \prod_\gamma^{-1}(U)$,
$y \in \prod_\gamma^{-1}(V) \,\&\, \prod_\gamma^{-1}(U) \cap \prod_\gamma^{-1}(V) = \prod_\gamma^{-1}(U \cap V) = \prod_\gamma^{-1} \emptyset = \emptyset$.

Definition 1.8

A function $f: (X,\tau) \longrightarrow (Y,\mu)$ is called

i- g - open function [**12**] if $f(g) \in go(X)$ for every $g \in \tau$.

ii-β- open function [**1**] if $f(G) \in \beta o(X)$ for every $G \in \tau$.

Definition 1.9

A subset A of a topological space (X,τ) is called β- ***generalized*** closed (briefly $g\beta$- ***closed***) if $\beta cl(A) \subseteq U$,whenever $A \subseteq U$ and $U \in \beta o(X)$.

Example 1.1

Let $X = \{a,b,c,d\} \,\&\, \tau = \{X, \emptyset, \{a\}, \{b\}, \{a,b\}, \{b,c,d\}\}$. Then we have
$\beta O(X) = \{\{b\}, \{a,b\}, \{b,c\}, \{b,d\}, \{a,b,c\}, \{a,b,d\}\}$,
$\beta C(X) = \{\{c\}, \{d\}, \{a,c\}, \{a,d\}, \{c,d\}, \{a,c,d\}\}$, $A = \{a,c\}$, $A \text{ is } g\beta$ - ***closed*** since $\exists\, U \in \beta O(X), U = \{a,b,c\}$, $\beta cl(A) \subseteq U, A \subseteq U$, also $B = \{d\}$ is $g\beta$ - ***closed***. But $C = \{a,c,d\}$ is not $g\beta$ - ***closed*** since $\nexists\, U \in \beta O(X), \ni \beta cl(A) \subseteq U$.

2 − $g\beta$-R_0 Spaces and $g\beta$-R_1 Spaces

In this section, we define and study two kinds of separation axioms
Namely, $g\beta$ - R_0 and $g\beta$ - R_1 spaces. Characterizations and properties of these spaces are provided.

Definition 2.1 [9]

If X is a topological space and $A \subset X$. Then the gb- $kernal$ of A (simply gb- $ker(A)$) is defined to be the set gb - $ker(A) = \cap\{U \in GBO(X), A \subseteq U\}$.

Definition 2.2

If X is a topological space and $A \subset X$. Then the $g\beta$- $kernal$ of A (simply $g\beta$-
$ker(A)$) is defined to be the set $g\beta$ - $ker(A) = \cap\{U \in G\beta O(X), A \subseteq U\}$.

Definition 2.3

We say that the family $G\beta O(X)$, has property (v) if the union of any collection of subsets belong to $G\beta O(X)$ in $G\beta O(X)$.

Definition 2.4

The $g\beta$- $closure$ (resp. $g\beta$- $interior$) of A, denoted by $g\beta cl(A)$ (resp. $g\beta int(A)$) is the intersection of all $g\beta$- $closed$ (resp. the union of all $g\beta$- $open$) sets containing A (resp. contained in A).

Definition 2.5

A sub set A of a topological space (X, τ) is said to be generalized β- open
(briefly $g\beta$- open) set if $U \subseteq \beta int(A)$ where ever $U \subseteq A$ and U is closed. The complement of generalized β- open set is said to be generalized β-closed. The family of all $g\beta$- open (resp $g\beta$- closed) sets of X is denoted by $G\beta O(X)$ (resp. $G\beta C(X)$).

Definition 2.6

We say that X is a $g\beta$-R_0 space if every $g\beta$-open set contains the $g\beta$- $cloure$ of each of its singletons.

Definition 2.7

We say that X is a $g\beta$-R_1 space if for any x, y in X with $g\beta cl(\{x\}) \neq g\beta cl(\{y\})$,
there exist disjoint $g\beta$- $open$ set U and Vsuch that $g\beta cl(\{x\})$ is a sub set of U and
$g\beta cl(\{y\})$ is a sub set of V.

Proposition 2.1

Let X be a space and $A \subset X$, then $x \in g\beta cl(\{A\})$ if and only if for each $g\beta$- $open$ set U containing x, $A \cap U \neq \emptyset$.

Proposition 2.2

Every $g\beta$- R_1 is $g\beta$- R_0.

Proof

Let U be a $g\beta$- $open$ set such that $x \in U$, if $y \notin U$, then $x \notin g\beta cl\{y\}$ and hence $g\beta cl\{x\} \neq g\beta cl\{y\}$. Then there is $g\beta$- $open$ set V such that $y \in V$ and $g\beta cl\{y\} \subset V$ and $x \notin V$, hence $y \notin g\beta cl\{x\}$. Then $g\beta cl\{x\} \subset U$. So X is a $g\beta$- R_0.

Theorem 2.1

A space X is a $g\beta$- R_0 space if and only if $y \in g\beta cl\{x\}$ for any two points x & y in X.

Proposition 2.3

If X is a topological space and x is any point in X. Then $y \in g\beta - ker(\{x\})$ if and only if $x \in g\beta c(\{y\})$.

Proof

Suppose $y \notin g\beta$- $ker(\{x\})$. Then there is a $g\beta$- $open$ set such that $x \in V$ & $y \notin V$. Then, we have $x \notin g\beta c(\{y\})$.Similarly, we can prove the converse.

Theorem 2.2

Let X be a space and A a subset of X and $G\beta O(X)$ has property (v) then $g\beta\, ker(A) = \{x \in X: g\beta cl(\{x\} \cap A) \neq \emptyset\}$.

Proposition 2.4

Let x, y be any two points in X, if $g\beta c(\{x\}) \neq g\beta c(\{y\})$, then
$g\beta\, ker(\{x\}) \neq g\beta\, ker(\{y\})$.

Proof

$\Rightarrow$ Suppose that $g\beta c(\{x\}) \neq g\beta c(\{y\})$, then there is a point z in X such that $z \in g\beta c(\{x\})$ and $z \notin g\beta c(\{y\})$. So there is a $g\beta$ - $open$ set U containing z and hence containing x but not containing y by (proposition 2.1). Then $y \notin g\beta ker(\{x\})$, there fore $g\beta ker(\{x\}) \neq g\beta ker(\{y\})$.

$\Leftarrow$ Assume that $g\beta ker(\{x\}) \neq g\beta ker(\{y\})$. Then there exist a point z in X such that $z \in g\beta ker(\{x\})$ but $z \notin g\beta ker(\{y\})$. Since $z \in g\beta ker(\{x\})$, then by (Theorem 2.2), $\{x\} \cap g\beta cl(\{z\}) \neq \emptyset$ and hence $x \in g\beta cl(\{z\})$. Since
$z \notin g\beta ker(\{y\})$, then $\{y\} \cap g\beta cl(\{z\}) = \emptyset$. And since $x \in g\beta cl(\{z\})$ and by assumption $g\beta cl(\{z\})$ is $g\beta$- $closed$ set. So $g\beta cl(\{x\}) \subset g\beta cl(\{z\})$ and then
$\{y\} \cap g\beta cl(\{z\}) = \emptyset$. Hence $g\beta cl(\{x\}) \neq g\beta cl(\{y\})$.

Proposition 2.5

If X is a topological space, and x is any point in X. Then
$y \in \cap\{U: U\, is\, g\beta o(X), A \subset U\}$ if and only if $x \in g\beta cl(\{y\})$.

Proof

Suppose that $y \notin \cap\{U: U\, is\, g\beta o(X), A \subset U\}$. Then there is a $g\beta$- $open$ set such that $x \in V$ and $y \notin V$. Then, we have $x \notin g\beta cl(\{y\})$. Similarly, we can prove the converse.

3 − $g\beta$- T_i Spaces, $i = 0,1,2$

In this section, we define and study a kind of separation axioms namely, $g\beta$-T_i,
$i = 0,1,2$ space. Characterizations and properties of the space is provided.

Definition 3.1

A space X is called generalized $g\beta$- T_0 space (briefly written as $g\beta$- T_0) If and only if to each pair of distinct points $x, y\, of\, X$, there exists a $g\beta$-$open$ set containing one but not the other.

Definition 3.2

A space X is called generalized $g\beta$- T_1 space (briefly written as $g\beta$- T_1) if and only if to each pair of distinct points $x, y\, of\, X$, there exists a pair of $g\beta$- $open$ sets, one containing x but not y, and the other containing y but not x.

Definition 3.3

A space X is called generalized $g\beta$- T_2 space (briefly written as $g\beta$- T_2) if
and only if to each pair of distinct points $x, y\, of\, X$, there exists a pair of disjoint $g\beta$- $open$ sets, one containing x, and the other containing y.

Remark 3.1

β- $T_2 \Rightarrow g\beta$- T_2 and $g\beta$- $T_2 \Rightarrow g\beta$- T_1 .
The converse of each part is not true in general.

Corollary 3.1

Every topological space X is $g\beta$- T_0.

Theorem 3.1

A space X is $g\beta$ - T_0 iff $\beta cl\{x\} \neq \beta cl\{y\}$ for every pair of distinct points

x, y of X.

Proof

Let x, y be any two points in X, such that $x \neq y$. Since every topological spas X is $g\beta$- T_0 space, by (Proposition 3.1) then there is a $g\beta$- open set U containing x or y, say x but not y. Then $X - U$ is a $g\beta$- closed contains y but does not contain x, and since $g\beta cl(\{y\}) \subseteq X - U$ then $x \notin g\beta cl(\{y\})$. There for $g\beta cl(\{x\}) \neq g\beta cl(\{y\})$.

Theorem 3.2

A space (X, τ) is $g\beta$- T_1 iff the singletons are $g\beta$- *closed* sets.

Proof

Suppose that X is $g\beta$- T_1 and x is any point in X, $y \in \{x\}^C$ then $x \neq y$. Then there exist a $g\beta$- *open* set U, such that $y \in U$, but $x \notin U$, Thus for each $y \in \{x\}^C$, there exist a $g\beta$- *ope* set U, such that $y \in U \subseteq \{x\}^C$. Then $\cup\{y: y \neq x\} \subseteq \cup\{U: y \neq x\} \subseteq \{x\}^C$, so that $\{x\}^C \subseteq \cup\{U: y \neq x\} \subseteq \{x\}^C$. Therefore $\{x\}^C = \{U: y \neq x\}$ since U is $g\beta$- *open* in X, by assumption. Hence $\{x\}^C$ is $g\beta$- *open* and so $\{x\}$ is $g\beta$- *closed*.

Theorem 3.3

For a space (X, τ), the following are equivalent

1- X is $g\beta$- T_2.

2- For each $x \in X$, $\cap\{g\beta cl(U): U \text{ is } g\beta - \text{open in } X \text{ containing } x\} = \{x\}$.

Proof

$1 \Rightarrow 2$ Let $, y \in X$ and $x \neq y$, then $y \notin \cap\{g\beta cl(U): U \in G\beta O(X), x \in U\} = \{x\}$.

Since X is $g\beta$- T_2 space, then there exist two disjoint $g\beta$- *open* sets U, V such that $x \in U, y \in V$. Then U^C is $g\beta$- *closed*, $y \notin g\beta cl(U)$. So $\cap\{g\beta cl(U): U \text{ is } g\beta - \text{open in } X \text{ and } x \in U\} = \{x\}$

$2 \Rightarrow 1$ Let $x \neq y$, then $y \notin \cap\{g\beta cl(U): U \in G\beta O(X), x \in U\} = \{x\}$. Hence there is a $g\beta$- *open* set V_Y containing y such that $V_Y \cap U = \emptyset$. There fore X is $g\beta$- T_2.

Theorem 3.4

For any space X, if X is $g\beta$- R_0 then X is $g\beta$- T_1.

Theorem 3.5

If $f: X \longrightarrow Y$ be an injective βg- *irresolute* mapping and Y is an $g\beta$- T_k. then X is $g\beta$- T_k, $(k = 0, 1, 2)$.

Proof

Obvious.

4 – β- Generalized Separation Axioms and β-Connectedness

In this section, we define a β- *separated* and we have provided some examples and theories that you achieve.

Definition 4.1 [20]

A topological space (X, τ) is said to be β-connected, if X cannot be expressed as a union of two non-empty and disjoint β- open subsets of (X, τ).

Definition 4.1[17]

A function $f: X \longrightarrow Y$ is said to be

1- β- continuous if the inverse image of each open set in Y is β- open X.

2- β- open if the image of each open set in X is -β open Y.

3- β- closed if the image of each closed set in X is β- closed Y.

Definition 4.1 [7]

Two subsets A *and* B in a space X are said to be b- *separated* if and only if $A \cap bcl(B) = \emptyset$ and $bcl(A) \cap B = \emptyset$.

Definition 4.2

Two subsets A *and* B in a space X are said to be β- *separated* if and only if $A \cap \beta cl(B) = \emptyset$ and $\beta cl(A) \cap B = \emptyset$.

Notice 4.1

From the fact that $\beta cl(A) \subset cl(A)$, for every sub set A of X, every separated set is β- *separated*. But the converse may not be true as shown in the following example.

Example 4.1

Let $X = \{a, b, c, d\}$ with a topology $\tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}$. The subset $\{a\}, \{c, d\}$ are β- *separated* but not *separated*

Example 4.2

Let $X = \{a, b, c, d\}$ with a topology $\tau = \{X, \emptyset, \{a\}, \{b, c\}, \{a, b, c\}\}$,

$\beta O(X) = \begin{Bmatrix} \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{b, d\} \\ \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\} \end{Bmatrix}$. The subsets $\{a\}, \{d\}$ are disjoint but not β- *separated*.

Theorem 4.1

Let A *and* B be nonempty sets in a space X then if A *and* B are β- *separated* and $U \subseteq A$ *and* $V \subseteq B$, then U *and* V are β- *separated*.

Proof

Since $U \subset A$ then $\beta cl(U) \subset \beta cl(A)$. Then $B \cap \beta cl(A) = \emptyset$, implies $V \cap \beta cl(A) = \emptyset$ *and* $V \cap \beta cl(U) = \emptyset$. Similarly $U \cap \beta cl(V) = \emptyset$. Hence U *and* V are β- *separated*.

Theorem 4.2

The sets A *and* B of a space X are β- *separated* if and only if there exist U *and* V in $BO(X)$ such that $A \subset U$, $B \subset V$ *and* $A \cap V = \emptyset, B \cap U = \emptyset$.

Proof

Let A *and* B be β- *separated* sets, $= X - \beta cl(A)$ *and* $U = X - \beta cl(B)$. Then $U, V \in BO(X)$ such that $A \subset U, B \subset V$ *and* $A \cap V = \emptyset$, $B \cap U = \emptyset$. On the other hand, let $U, V \in BO(X)$ such that $U, V \in BO(X)$. Since $X - V$ *and* $X - U$ are β- *closed*, then $\beta cl(A) \subset X - V \subset X - B$ and $\beta cl(B) \subset X - U \subset X - A$. Then $\beta cl(A) \cap B = \emptyset$ *and* $\beta cl(B) \cap A = \emptyset$.

Conclusion

The class of generalized closed sets has an important role in general topology, especially its suggestion of new separation axioms which are useful in digital topology. Separation axioms in terms of $g\beta$- open sets have been formulated and their structural properties have also been discussed. In this work we introduced and study new types of separation axioms namely, $g\beta$- R_i, i = 0, 1, and $g\beta$ - T_i Ti, i = 1,2. Several characterizations and properties of these concepts are provided.

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