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On Oscillatory Solutions of Neutral Functional Differential Equations

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A B S T R A C T

In this article, a specific neutral functional differential equations class is considered. Some new sufficient conditions related to the oscillatory solutions are constructed. These conditions extend and improve several well-known conditions in many existing studies.

عن الحلول المتذبذبة للمعادلات التفاضلية الدالية المحايدة

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الكلمات المفتاحية:
الحلول المتذبذبة
المحايدة
المعادلات التفاضلية الدالية

الملخص

في هذه الورقة، نتناول فصل محدد من المعادلات التفاضلية الدالية المحايدة. نصيغ بعض الشروط الجديدة الكافية المتعلقة بالحلول المتذبذبة. تُوسِّع هذه الشروط وتُحسِّن العديد من الشروط المعروفة في العديد من الدراسات الموجودة حاليا..

1. Introduction

In addition of its theoretical interest, the theory of oscillations in neutral differential equations posses several interesting applications. For instance, it can be applied to areas such as population growth, the motion of radiating electrons, the spread of epidemics etc. (see Driver [10]; Gyori & Ladas [16]; Hale [17]; Onose [20]).

The first work demonstrating a criterion for oscillatory solutions of a neutral equations was proved and published by Zahariev and Bainov [28]. Since then, the oscillation theory of neutral functional differential equations has been studied and developed intensively. For example, one can see in the papers of Ahmed et al. [2]-[6], Moaaz and Al-Jaser [19], Farrel et al. [12], Saker and Kubiaczyk [21], Santra [22], and Zhou [29] as well as the monographs by Agarwal et al. [1], and Gyori and Ladas [16]. For more discussions of the applications of these equations, as well as how the behaviour of their solutions, which differs from that of non-neutral ones, the readers are referred to the papers of Lalli and Zhang [18], Stavroulakis [24], and Sficas and Stavroulakis [23].

Consider the neutral delay differential equation of the form

$$\big(a(t)x(t)-p(t)x(t-\tau)\big)'+q(t)x(t-\sigma)=0 \qquad (1.1)$$

where

$$a,p,q \in C\big[[t_0,\infty[,(0,\infty)\big],\ \tau,\sigma>0 \qquad (1.2)$$

Recently, there has been considerable research development focused on the oscillation (and nonoscillation) of neutral differential equations. Several authors including Farrel et al. [12], Yu et al. [26], Grammatikpoulos et al. [13], Greaf et al. [14], Chuanxi et al. [9] and Chuanxi and Ladas [8] have investigated oscillations of solutions of different classes of Eq. (1.1). In general, all the papers mentioned above except Yu et al. [26] always give the oscillatory conditions under the condition

$$\int_{t_0}^{\infty} q(t)dt=\infty, \qquad (1.3)$$

which is an essential condition for the oscillation. Naturally, one might wonder if condition (1.3) is a necessary requirement to the oscillation of all solutions related to Eq. (1.1). For the case when $a(t)\equiv 1$ and $p(t)\equiv 1$, Yu et al. [26] relaxed the condition (1.3) to the condition

$$\int_{t_0}^{\infty} sq(s)\big(\int_{s}^{\infty} q(u)\,du\big)ds=\infty, \qquad (1.4)$$

which answered the question proposed by Chuanxi and Ladas [8].

Chen et al. [7] considered Eq. (1.1) in the case $a(t)\equiv 1$, and established some related new sufficient conditions of the oscillation without the assumptions (1.3) and (1.4). Following our previous work (Ahmed et al. [2-6]), we are still interested in obtaining some further results in the study of oscillatory theory to the neutral differential equations. For further samples of oscillation criteria, see recent papers by Elabbasy et al. [11], Yu et al. [25] and Yu [27]. In this paper, our primary goal is to apply the results of Chen et al. [7] to Eq. (1.1) in order to find new conditions under which all solutions of Eq. (1.1) oscillate. This will allow us to generalize and extend some previous result.

Defination 1.1 A solution of Eq. (1.1) is called oscillatory if it has arbitrarily large zeros.

Defination 1.2 Equation (1.1) is considered oscillatory, if all of its solutions are oscillatory.

Before formulating the main results, we present the following theorem that plays an essential role in the proofs of our main results.

Theorem 1.1 Yu et al. [26]: Assume that condition (1.4) holds. Then

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every solution of the equation

$$\left(x(t)-x(t-\tau)\right)'+q(t)x(t-\sigma)=0 \qquad (1.5)$$

oscillates.

2. Main Results

The following lemma is a quite useful element in the proofs of our new results.

Lemma 2.1: Suppose the condition (1.2) is satisfied, and let that there exists $t^* \geq t_0 > 0$ such that

$$\frac{p(t^*+n^*\tau)}{a(t^*+(n^*-1)\tau)} \leq 1,\ \ n^*=0,1,2,3,.... \qquad (2.1)$$

where $x(t)$ is an eventually positive solution of Eq. (1.1). Define

$$z(t)=a(t)x(t)-p(t)x(t-\tau) \qquad (2.2)$$

Then

$$z(t)>0,\ z'(t)\leq 0.$$

Proof. See Saker and Kubiaczyk [21].

Theorem 2.1: Suppose that the conditions (1.2) and (2.1) are satisfied. Assume that either

$$p(t)+\sigma q(t)>0 \qquad (2.3)$$

or

$$\sigma>0 \text{ and } q(t) \text{ is not identically zero for every } s\in[t,t+\sigma] \qquad (2.4)$$

Then, every solution of Eq. (1.1) oscillates if and only if the associated inequality:

$$\left(a(t)x(t)-p(t)x(t-\tau)\right)'+q(t)x(t-\sigma)\leq 0 \qquad (2.5)$$

has no eventually positive solution.

Proof. Let $x(t)$ be an eventually positive solution of the inequality (2.5). We plan and aim to show that Eq. (1.1) has a nonoscillatory solution. Let $z(t)$ be defined as in (2.2). Then according to Lemma 2.1, we have

$$z(t)\geq\int_t^\infty q(s)x(s-\sigma)ds \qquad (2.6)$$

That's

$$x(t)\geq\frac{1}{a(t)}\left[p(t)x(t-\tau)+\int_t^\infty q(s)x(s-\sigma)ds\right] \qquad (2.7)$$

Let $T>t_0$ be fixed so that (2.7) holds for all $t\geq T$. Set $T_0\in max\{\tau,\sigma\}$ and then, consider the set of functions

$$X=\{u\in C[(T-T_0,\infty),\mathbb{R}^+];\ \ 0<u(t)<1,\ \ t\geq T-T_0\}$$

Define a mapping F on X as

$$(Fu)(t) = \begin{cases}\dfrac{1}{a(t)x(t)}\left[p(t)u(t-\tau)x(t-\tau)+\displaystyle\int_t^\infty q(s)u(t-\sigma)x(s-\sigma)ds\right], & t\geq T\\ \dfrac{t-T+T_0}{T_0}(Fu)(T)+\left(1-\dfrac{t-T+T_0}{T_0}\right), & T-T_0\leq t\leq T\end{cases}$$

It is straightforward to observe, by utilizing (2.7), where F maps X into itself. Furthermore, for any $u\in X$, we have

$$(Fu)(t)>0, \qquad T-T_0\leq t\leq T$$

In the next, the sequence $u_i(t)$ in X be defined as in the following manner:

$$u_0(t)=1,\ \ t\geq T-T_0\ ,$$
$$u_{i+1}(t)=(Fu_i)(t),\ \ i=0,1,2\,...$$

Therefore, by using (2.7) and a simple induction, we can easily see that

$$0\leq u_{i+1}(t)\leq u_i(t)\leq 1,\ t\geq T-T_0,\ \ i=0,1,2,...$$

Let

$$u(t)=\lim u_i(t)\,,\ t\geq T-T_0,\ \ i=0,1,2,...$$

Therefore, it follows from the Lebesgue's dominated convergence theorem that, the $u(t)$ satisfies

$$u(t)=\left(\frac{1}{a(t)x(t)}\right)\left(p(t)u(t-\tau)x(t-\tau)+\int_t^\infty q(s)u(s-\sigma)\,x(s-\sigma)ds\right),\ \ t\geq T$$

and

$$u(t)=\left(\frac{t-T+T_0}{T_0}\right)(Fu)(T)+\left(1-\frac{t-T+T_0}{T_0}\right),\ \ \ T-T_0\leq t\leq T$$

Let

$$\omega(t)=a(t)u(t)x(t)$$

Then, we have

$$\omega(t)>0,\ \ \ T-T_0\leq t<T$$

and satisfies for $t\geq T$ the following equality,

$$\omega(t)=\left[\left(\frac{1}{a(t)x(t)}\right)\left(p(t)u(t-\tau)x(t-\tau)+\int_t^\infty q(s)u(s-\sigma)\,x(s-\sigma)ds\right)\right]\times\left(\frac{1}{a(t)x(t)}\right),$$

which implies that

$$\omega(t)=\bar{p}(t)\omega(t-\tau)+\int_t^\infty\bar{q}(s)\omega(s-\sigma)\,ds, \qquad t\geq T \qquad (2.8)$$

where

$$\bar{p}(t)=\frac{p(t)}{a(t-\tau)} \text{ and } \bar{q}(s)=\frac{q(s)}{a(s-\sigma)}$$

Clearly, $\omega(t)$ is a continuous function on $t\geq T-T_0$.

To demonstrate that $\omega(t)$ is positive on $t\geq T-T_0$, we suppose that there exists some $t^*\geq T-T_0$, where

$$\omega(t)>0 \text{ for all } T-T_0\leq t< t^* \text{ and } \omega(t^*)=0$$

Then $t^*\geq T$, where according to (2.8), yields

$$0=\bar{p}(t^*)\,\omega(t^*-\tau)+\int_{t^*}^\infty\bar{q}(s)\omega(s-\sigma)\,ds, \qquad t^*\geq T$$

So,

$$p(t^*)=0 \text{ and } q(s)\,\omega(s-\sigma)=0 \text{ for all } s\geq t^*$$

This contradicts (2.3) or (2.4). Therefore, $\omega(t^*)$ is positive on $(T-T_0,\infty)$. Furthermore, it is easy to see that $\omega(t)$ is a positive solution of Eq. (1.1). It implies that (2.5) having no eventually positive solution is a necessary condition for the oscillation of Eq. (1.1). The proof is complete.

Remark 2.1: Theorem 2.1 extends Theorem 1 in Chen et al. [7] and extends and improves Theorem 2.1 in Lalli and Zhang [18] by removing the infinite integral condition (1.3).

In what follows, we present some applications of Theorem 2.1.

Theorem 2.2: Assume that (1.2) and (2.1) hold. Suppose that

$$\bar{q}(t)p(t-\sigma)\geq q(t-\tau) \qquad (2.9)$$

and

$$\int_{t_0}^\infty s\bar{q}(s)\left(\int_s^\infty\bar{q}(u)\,du\right)ds=\infty \qquad (2.10)$$

where,

$$\bar{q}(t)=\frac{q(t)}{a(t-\sigma)}$$

Then, every solution of Eq. (1.1) is oscillatoty.

Proof. Suppose, for contradiction, that Eq. (1.1) has an eventually positive solution $x(t)$. Let $x(t)>0$ for all $t\geq t_0>0$ and $z(t)$ be defined as in (2.2). Then, according to Lemma 2.1, we obtain

$$z(t)>0 \text{ and } z'(t)\leq 0$$

From Eq. (1.1), we have

$$x(t)=\frac{1}{a(t)}\left(z(t)+p(t)x(t-\tau)\right), \qquad (2.11)$$

and

$$x(t-\sigma)=\frac{1}{a(t-\sigma)}\left(z(t-\sigma)+p(t-\sigma)x(t-\sigma-\tau)\right), \qquad (2.12)$$

Hence, from (1.1), (2.12) and (2.9), we obtain,

$$\begin{aligned} z'(t)&=-q(t)x(t-\sigma)\\ &=-q(t)\left(\frac{1}{a(t-\sigma)}\left(z(t-\sigma)+p(t-\sigma)x(t-\sigma-\tau)\right)\right)\\ &\leq\frac{-q(t)}{a(t-\sigma)}z(t-\sigma)-q(t-\tau)x(t-\sigma-\tau)\end{aligned}$$

That's

$$z'(t)\leq\frac{-q(t)}{a(t-\sigma)}z(t-\sigma)+z'(t-\tau)$$

or

$$z'(t)-z'(t-\tau)+\bar{q}(t)z(t-\sigma)\leq 0, \qquad (2.13)$$

where

$$\bar{q}(t)=\frac{q(t)}{a(t-\sigma)}$$

By Theorem 2.1 with the inequality (2.13), we have

$$z'(t)-z'(t-\tau)+\bar{q}(t)z(t-\sigma)=0 \qquad (2.14)$$

has an eventually positive solution. However, according to Theorem 1.1, condition (2.10) shows that Eq. (2.14) cannot possess an eventually positive solution, which leads to a contradiction. The proof is complete.

Theorem 2.3: Let (1.2) and (2.10) be satisfied. Suppose also that

$$0 < a(t) \le 1,\ \ p(t) \ge 1 \tag{2.15}$$

and

$$\bar{q}(t)p(t-\sigma) \le q(t-\tau). \tag{2.16}$$

Then every solution of Eq. (1.1) is oscillatory.

Proof. Assume that Eq. (1.1) has an eventually positive solution $x(t)$. The case where $x(t)$ is eventually negative is analogous and will be omitted. Let $z(t)$ is defined as in (2.2). Then

$$z(t) < 0 \text{ and } z'(t) \le 0$$

From equations (1.1) and (2.12) and condition (2.16), it follows that

$$z'(t) - z'(t-\tau) + \bar{q}(t)z(t-\sigma) \ge 0 \tag{2.17}$$

Therefore, $-z(t)$ is an eventually positive solution of the following inequality

$$z'(t) - z'(t-\tau) + \bar{q}(t)z(t-\sigma) \le 0 \tag{2.18}$$

Therefore, by Theorem 2.1 and Theorem 1.1, we are led to a contradiction which completes the proof.

Example 2.1: Consider the equation

$$\left(x(t) - \frac{t+e}{t}x(t-e)\right)' + \frac{2}{5\sqrt{t^3}}x(t-e) = 0,\ \ t \ge \pi \tag{2.19}$$

Here we have

$$a(t) = 1,\ \ p(t) = \frac{t+e}{t},\quad q(t) = \frac{2}{5\sqrt{t^3}},\quad \tau = \sigma = e$$

Observe also that

$$\bar{q}(t)p(t-\sigma) = \frac{2}{5\sqrt{t^3}} \times \frac{t}{t-e} \le \frac{2}{5\sqrt{(t-e)^3}} = q(t-\tau)$$

and

$$\int_{t_0}^{\infty} s\bar{q}(s)\left(\int_s^{\infty}\bar{q}(u)\,du\right)ds = \frac{4}{25}\int_{\pi}^{\infty}\frac{s}{\sqrt{s^3}}\left(\int_s^{\infty}\frac{1}{\sqrt{u^3}}du\right)ds$$

$$= \frac{8}{25}\int_s^{\infty}\frac{ds}{s} = \infty$$

That is, all conditions of Theorem 2.3 are satisfied and therefore every solution of Eq. (2.19) is oscillatory.

Remark 2.2: Theorems 2.2 and 2.3 extend results of Chen et al. [7], Yu et al. [25] and Lalli and Zhang [18]. Also, one can see the results of Saker and Kubiaczyk [21], Ahmed et. al. [5] and Ahmed et. al [6]**.**

3. Conclusion

In conclusion, a neutral differential equations class is studied and a set of new oscillation theorems are stated and proved. Our results improve, enhance and generalize many of the well-known results in the literature.

4. Competing Interests:

The authors declare that they have no competing interests.

5. Authors' Contributions

Prof. Souad A. Abumaryam contributed in drafting this manuscript, Prof. Fatima N. Ahmed, the corresponding author, has given the main proofs and Prof. Ioannis P. Stavroulakis reviewed the whole manuscript and improved it to come out with this revised draft. All authors read and approved the final manuscript.

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7. References

[1]- Agarwal, R. P., Bohner, M. & Li, W. T. 2004. Nonoscillation and oscillation: Theory for functional differential equations. New York Marcel Dekker.

[2]- Ahmed, F. N., Ahmad, R. R., Din, U. K. S. & Noorani, M. S. M. 2013. Oscillation Criteria of First Order Neutral Delay Differential Equations with Variable Coefficients. Abstract and Applied Analysis, Vol. 2013, Article ID 489804, 5 pages.

[3]- Ahmed, F. N., Ahmad, R. R., Din, U. K. S. & Noorani, M. S. M. 2014. Some further results on oscillations for neutral delay differential equations with variable coefficients. Abstract and Applied Analysis, Vol. 2014, Article ID 579023, 6 pages.

[4]- Ahmed, F. N., Ahmad, R. R., Din, U. K. S. & Noorani, M. S. M. 2014. Oscillations for Nonlinear Neutral Delay Differential Equations with Variable Coefficients. Abstract and Applied Analysis, Vol. 2014, Article ID 179195, 6 pages.

[5]- Ahmed, F. N., Ahmad, R. R., Din, U. K. S. & Noorani, M. S. M. 2015. Oscillation criteria for nonlinear functional differential equations of neutral type. J. of Inequalities and Applications. DOI 10.1186/s13660-015-0608-5, 2015:97.

[6]- Ahmed, F. N., Ahmad, R. R., Din, U. K. S. & Noorani, M. S. M. 2018. Some open problems in the theory for neutral delay differential equations, Scientific Journal of Applied Sciences. 8: 12-18.

[7]- Chen, M. –P., Yu, J. S. & Huang, L. H. 1994. Oscillations of first order neutral differential equations with variable coefficients. J. of Math. Anal. Appl. 185:288-301.

[8]- Chuanxi, Q. & Ladas, G. 1989. Oscillations of neutral differential equations with variable coefficients. Appl. Anal. 32: 215-228.

[9]- Chuanxi, Q., Ladas, G., Zhang B. G. & Zhao, T. 1990. Sufficient conditions for oscillation and existence of positive solutions, Appl. Anal. 35: 187-194.

[10]- Driver, R. D. 1984. A mixed neutral system. Nonlinear Analysis. 8: 155-158.

[11]- Elabbasy, E. M., Hassan T. S. & Saker, S. H. 2005. Necessary and sufficient condition for oscillations of neutral differential equation. Serdica Math. J. 31: 279-290.

[12]- Farrell, K., Grove, E. A. & Ladas, G. 1988. Neutral delay differential equations with positive and negative coefficient. Applied Analysis. 27: 181-197.

[13]- Grammatikopoulos, M. K., Grove, E. A. & Ladas, G. 1986. Oscillations of first order neutral delay differential equations. J. Math. Anal. Appl. 120: 510-520.

[14]- Graef, J. R., Savithri R. & Thandapani, E. 2004. Oscillation of first order neutral delay differential equations, EJQTDE. No. 12: 1-12.

[15]- Graef, J. R., Grammatikopoulos, M. K. & Spikes, P. W. 1991. Asymptotic and oscillatory behaviour of solutions of first order nonlinear delay differential equations. J. Math. Anal. Appl. 155: 562-571.

[16]- Gyori I. & Ladas, G. 1991. Oscillation theory of delay differential equations. Oxford Mathematical Monographs. New York, Clarendon Press.

[17]- Hale, J. K. 1977. Theory of functional differential equations. New York, Springer.

[18]- Lalli, B. S. & Zhang, B. G. 1990. Oscillation of first order neutral differential equations. Appl. Anal. 39: 265-274.

[19]- Moaaz, O. and Al-Jaser, A. 2024. Functional differential equations of the neutral type: Oscillatory features of solutions. AIMS Mathematics, 9(6):16544-16563.

[20]- Onose, H. 1983. Oscillatory properties of the first order differential inequalities with deviating argument. Funkcialaj Ekvacioj. 26: 189-195.

[21]- Saker, S. H. & Kubiaczyk, I. 2002. Oscillation of solutions to nonlinear neutral delay differential equations. J. Appl. Anal. 8: 261-278.

[22]- Santra, Sh. S. 2016. Existence of positive solution and new oscilaation criteria for nonlinear first-order neutral delay differential equations. J. of Differential Equations & Applications. 8: 33-51.

[23]- Sficas, Y. G. & Stavroulakis, I. P. 1987. Necessary and sufficient conditions for oscillations of neutral differential equations. J. Math. Anal. Appl. 123:494-507.

[24]- Stavroulakis, I. P. 1989. Oscillation of mixed neutral equations. Hiroshima Math. J. 19: 441-456.

[25]- Yu, J. S., Chen, M. –P. & Zhang, H. 1998. Oscillation and nonoscillation in neutral equations with integrable coefficients. Computers Math. Applic. 35: 65-71.

[26]- Yu, J. S., Wang, Z. C. & Chuanxi, Q. 1992. Oscillation of neutral delay differential equations. Bull. Austral. Math. Soc. 45: 195-200.

[27]- Yu, Y. -H. 1991. Oscillation of solutions of neutral delay differential equations. Acta. Appl. Math. Sinica. 14: 404-410.

[28]- Zahariev, A. I. & Bainov, D. D. 1980. Oscillating properties of the solutions of a class of neutral type functional differential equations. Bull. Austral. Math. Soc. 22: 365-372.

[29]- Zhou, Y. 2000. Oscillation of neutral functional differential equations. Acta. Math. Hungar. 86: 205-212.