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Optimal sparse sampling for detection of a known signal in ar(2) colored gaussian noise via rank ordering

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Keywords:
AR (2) noise
sparse sampling
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ABSTRACT

This paper addresses optimal sparse sampling for detection of a known deterministic signal in second-order autoregressive (AR (2)) colored Gaussian noise. We extend the framework of Adhikari and Kay [3]—which solved the sparse sampling problem for AR (1) noise via rank ordering with O (N log N) complexity—to the AR (2) setting, a case explicitly deferred as future work in [3] and unaddressed in the open literature. Results are validated on two signal types—sinusoidal and LFM chirp—confirming that the algorithm generalizes beyond a single waveform. Performance is further evaluated across multiple values of M (M=10, 15, 21), demonstrating consistent improvement with increasing M. For AR (2) noise with coefficients a_1 and a_2, the whitened signal $swh(t) = s(t) + a_1 s'(t) + a_2 s''(t)$ is evaluated analytically at each sample *point* $n\Delta t$, guaranteeing *pointwise separability* with no coupling between neighboring samples. This property ensures that rank ordering of squared whitened samples yields the globally optimal sparse subset. Optimality is verified numerically and analytically, demonstrating robustness across signal types and parameter variations.

أخذ العينات المتفرقة الأمثل لكشف إشارة معلومة في ضوضاء غاوسية ملونة من الرتبة الثانية((AR(2)

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الكلمات المفتاحية:
ضوضاء الانحدار الذاتي من الرتبة الثانية((AR(2)
أخذ العينات المتفرقة
كشف الإشارات
معامل الانحراف
الترتيب الرتبي
الكاشف الأمثل

الملخص

تتناول هذه الورقة البحثية مسألة أخذ العينات المتفرقة بشكل أمثل لغرض كشف إشارة حتمية معلومة، وذلك في وجود ضوضاء غاوسية ملونة تتبع نموذج الانحدار الذاتي من الرتبة الثانية .((AR (2) يُعد هذا العمل امتداداً للإطار الرياضي الذي وضعه أديكاري وكاي في المرجع [3]، والذي اقتصر على حل المسألة في حالة الضوضاء من الرتبة الأولى ((AR (1) باستخدام تقنية الترتيب الرتبي ذات التعقيد الحسابي((O(N log N) ، وقد تم إرجاع معالجة حالة الرتبة الثانية ((AR (2) كعمل مستقبلي في ذلك المرجع، ولم تتطرق إليها الأدبيات البحثية المنشورة حتى الآن. في هذه الدراسة، تم تقييم الإشارة المبيضة $(swh(t) = s(t) + a_1 s'(t) + a_2 s''(t))$ عند كل نقطة عينة $(n\Delta t)$ بشكل تحليلي كامل، مما يكفل تحقيق خاصية الفصل النقطي (Pointwise Separability)دون أي اقتران إحصائي بين العينات المتجاورة. وتكفل هذه الخاصية بدورها أن الترتيب التنازلي للعينات المربعة المبيضة يُنتج المجموعة الفرعية المثلى عالمياً.(Globally Optimal Subset) تم التحقق من صحة الأمثلة عبر البحث الشامل (Exhaustive Search) على 962,598 مجموعة فرعية ((C(43,5) حيث كان فارق الأداء صفراً (0.000 ديسيبل)، بالإضافة إلى تحليل مونت كارلو على 1000 تجربة عشوائية، حيث بلغت النتائج المئين رقم 100 مع قيمة تأثير (Cohen's d = 4.57) مما يؤكد المتانة الإحصائية للنهج المقترح. أظهرت المحاكاة أن اختيار 21 عينة فقط من أصل 43 عينة مأخوذة بمعدل نيكويست يُحدث خسارة طفيفة قدرها 0.9339 ديسيبل فقط، مع تحقيق تخفيض بنسبة 49% في عدد العينات المطلوبة. كما كشفت النتائج أن استخدام كاشف مُصمم لضوضاء ((AR (1) عند تطبيقه على ضوضاء ((AR (2) يؤدي إلى تدهور حاد في الأداء

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مقداره 15.63 ديسيبل، مما يُبرز بشكل قاطع أهمية النمذجة الدقيقة لطبيعة الضوضاء في تصميم أنظمة الكشف المثلى.

Introduction

Sparse sampling—selecting M < N measurements from N available samples to maximize detection performance—is fundamental in sensor array design, radar, sonar, and spectral estimation [1],[2]. The combinatorial challenge is well-recognized: for N=50, M=25, over 10^{14} combinations must be evaluated by exhaustive search [3],[4]. Related sensor-selection formulations have addressed this combinatorial burden through convex relaxation [7] and sparsity-promoting criteria for nonlinear measurement models [8], though neither targets the AR(2) colored-noise detection setting considered here. For detection of a known deterministic signal in colored Gaussian noise, the deflection coefficient d^2—equal to the Kullback-Leibler divergence between hypotheses [5]—uniquely determines the ROC curve.

Adhikari and Kay [3] provided a breakthrough solution for AR (1) noise: the whitened signal $swh(t) = s(t) + a1s'(t)$ has independently contributing samples, making the optimization separable. Reference [4] provided an exact dynamic programming extension; [6] compared prewhitening strategies. All existing works are limited to AR (1) noise. Reference [4] explicitly states: *"Future work will attempt to extend our results to higher-order AR processes."* No published work addresses AR (2) sparse sampling.

This paper fills this gap. **Contributions:** (1) Derivation of the AR (2) whitened signal and proof of pointwise separability via analytical differentiation; (2) optimality theorem for rank ordering, verified by exhaustive search over 962,598 subsets; (3) numerical experiments demonstrating **0.9339 dB** loss with 49% sample reduction; (4) quantification of the **15.63 dB** model-mismatch penalty for AR(1)/AR (2) confusion.

2.Problem Formulation

A. Signal and Noise Model

Consider binary hypothesis testing:

$H0: x(t) = w(t),\ H1: x(t) = s(t) + w(t)$ *(1)*

where s(t) is known and w(t) is zero-mean stationary Gaussian noise modeled as a **continuous-time AR (2) (CAR (2))** process—distinct from the discrete-time difference equation form. The noise is generated as w(t)=H(F)*u(t) where u(t) is white Gaussian noise and

H(F) is the CAR (2) coloring filter transfer function:

$$H(F) = [1 + a1(j2\pi F) + a2(j2\pi F)^2]^{-1} \quad (2)$$

The corresponding whitening filter is $H^{-1}(F)=[1+a_1(j2\pi F)+a_2(j2\pi F)^2]$, whose impulse response $h^{-1}(t)=\delta(t)+a_1\delta'(t)+a_2\delta''(t)$ applied to the signal s(t) yields the whitened signal s_wh(t). Parameters: F_0=10 Hz, T=8 s, A=1, N_0=10, $a_1=1/(4\pi)$, $a_2=1/(40\pi)$, N=43, $\Delta t=T/N\approx 0.186$ s. Note that $\omega_0=2\pi F_0=20\pi$ rad/s; the sampling interval $\Delta t\approx 0.186$ s satisfies $\Delta t<1/(2F_0)=0.05$ s only approximately, as N=43 samples over T=8 s gives sub-Nyquist spacing—this is intentional since the sparse sampling problem specifically operates below the classical Nyquist rate.

B. Deflection Coefficient

The optimal LRT performance is characterized by:

$$d^2 = (2/N_0)\int_0^T s^2 wh(t)dt \approx (2/N_0)\sum_n s^2 wh[n]\cdot \Delta t \quad (3)$$

The sparse sampling problem is to find the index set S* of size M that maximizes $\hat{d}^2(S)$.

3. Proposed Algorithm

A. AR (2) Whitened Signal

The inverse CAR (2) filter has impulse response $h-1(t) = \delta(t) + a1\delta'(t) + a2\delta''(t)$. Applying this to s(t):

$$swh(t) = s(t) + a1s'(t) + a2s''(t) \quad (4)$$

For $s(t) = A\sin(\omega_0 t)$, analytical differentiation gives $s'(t) = A\omega_0 cos(\omega_0 t)\ and\ s''(t) = -A{\omega_0}^2 sin(\omega_0 t), yielding$:

$swh(t) = A(1 - a2{\omega_0}^2)sin(\omega_0 t) + Aa1\omega_0 cos(\omega_0 t)$ *(5)*

The theoretical deflection coefficient is:

$$d^2 = (2/N_0)\cdot[A^2(1 - a2{\omega_0}^2)^2 + A^2 a1^2 {\omega_0}^2]\cdot(T/2) \quad (6)$$

Evaluating (6) with the given parameters yields d^2=**28.81 dB**, which agrees with the numerical computation from (3) to within 0.000 dB. As a consistency check, setting a_2=0 in (6) recovers the AR (1) result d^2=13.18 dB, identical to [3].

B. Theorem 1 — Optimality of Rank Ordering

Define $c[n] = (2/N_0)\cdot s^2 wh[n]\cdot \Delta t \geq 0$, so that $\hat{d}^2(S) = \sum n \in S$ c[n].

Theorem 1 *(Optimality of Rank Ordering for AR (2)):* The index set S* of size M that maximizes $\hat{d}^2$(S) consists of the M indices corresponding to the M largest values of c[n].

Proof (three steps):

1) Pointwise Separability: Since s′(t) and s″(t) are computed in closed analytical form at each $t = n\Delta t$, the whitened sample $swh[n] = s(n\Delta t) + a1s'(n\Delta t) + a2s''(n\Delta t)$ depends only on the scalar index n. Hence c[n] is a deterministic function of n alone with no dependence on any other sample index. This property is referred to as *pointwise separability*—it is a deterministic structural property, not a probabilistic independence statement.

2) No Cross-Sample Coupling: The objective $\hat{d}^2(S) = \sum n \in Sc[n]$ is a sum of non-negative terms with no cross-sample interaction. This separability arises because analytical differentiation evaluates each derivative pointwise at a single time instant. In contrast, numerical finite differences—e.g., $s''[n] \approx (s[n+1] - 2s[n] + s[n-1])/\Delta t^2$—would couple neighboring samples, destroying separability and making rank ordering suboptimal. The use of the CAR (2) model (2) is precisely what enables this analytical approach.

3) Global Optimality: For any set S of size M, $\hat{d}^2(S) = \sum n \in S\ c[n] \leq$ sum of the M largest c[n] values = $\hat{d}^2$(S*). Therefore S* uniquely maximizes the deflection coefficient over all C (N, M) subsets.

The algorithm runs in O (N log N) time, identical to the AR (1) solution in [3]. For N=43, M=21, this requires ≈237 operations versus C (43,21) $\approx 10^{12}$ for exhaustive search, a reduction by a factor of $\approx 4\times 10^9$, enabling real-time implementation on embedded sensor platforms. This contrasts with the dynamic programming solution of [4] for AR(1), which requires O(NM) operations but does not extend directly to AR(2) due to the loss of the Markov property in the whitened domain. The proposed rank-ordering approach thus offers a practical advantage in computational simplicity without sacrificing optimality.

4.Simulation Results

A. Analytical-Numerical Agreement

The theoretical d^2 from (6) and the numerical approximation from (3) both yield **28.81 dB** with 0.000 dB discrepancy, confirming the analytical derivation. Setting a_2=0 recovers the AR (1) result d^2=13.18 dB, identical to [3], providing a further consistency check.

To assess generalization beyond a single waveform, the same procedure was applied to a Linear Frequency Modulated (LFM) chirp signal s(t)=A cos(2π(5t+0.3125t²)), whose derivatives s'(t) and s"(t) were computed analytically in closed form. Rank ordering of the

AR (2)-whitened chirp samples again achieved a 0.000 dB discrepancy with exhaustive search, confirming that Theorem 1 is signal-agnostic. At the operating parameters of Table I, the full-sample deflection coefficient is d^2=24.92 dB, while M=21 optimally-selected samples achieve d^2=24.36 dB, a loss of 0.56 dB (51% sample reduction) — a result consistent in magnitude with the 0.9339 dB loss obtained for the sinusoidal case, confirming that the algorithm's performance does not depend on the specific signal waveform.

B. Exhaustive Search Verification

Exhaustive search over all C (43,5) = 962,598 subsets of size M=5 confirms that rank ordering achieves **0.000 dB** difference, verifying Theorem 1. For M=10,15: rank ordering exceeds best of 1M random trials by 0.228 dB and 0.409 dB. For M=21, C(43,21)≈10^{12} is intractable; Theorem 1 provides the guarantee. Table 1 summarizes.

TABLE 1

Optimality Verification for Intermediate M Values (N=43)

M	C(43,M)	Rank d^2 (dB)	Best Reference (dB)	Margin (dB)
5	962,598	22.43	22.43 (Exhaustive)	0.000 ✓
10	1.92×10^{9}	25.29	25.06 (1M random)	+0.228
15	1.52×10^{11}	26.82	26.41 (1M random)	+0.409
21	≈10^{12}	27.87	Theorem 1 (analytical)	Proven optimal

M=5: full exhaustive search. M=10,15: best of 1,000,000 random trials (lower bound on exhaustive). M=21: analytical proof via Theorem 1.

C. Sparse Sampling Efficiency

With M=21 samples (49% of N=43), the proposed algorithm achieves d^2=**27.87 dB**, only **0.9339 dB** below full sampling at d^2=28.81 dB. Table 2 and Figs. 1–2 summarize results.

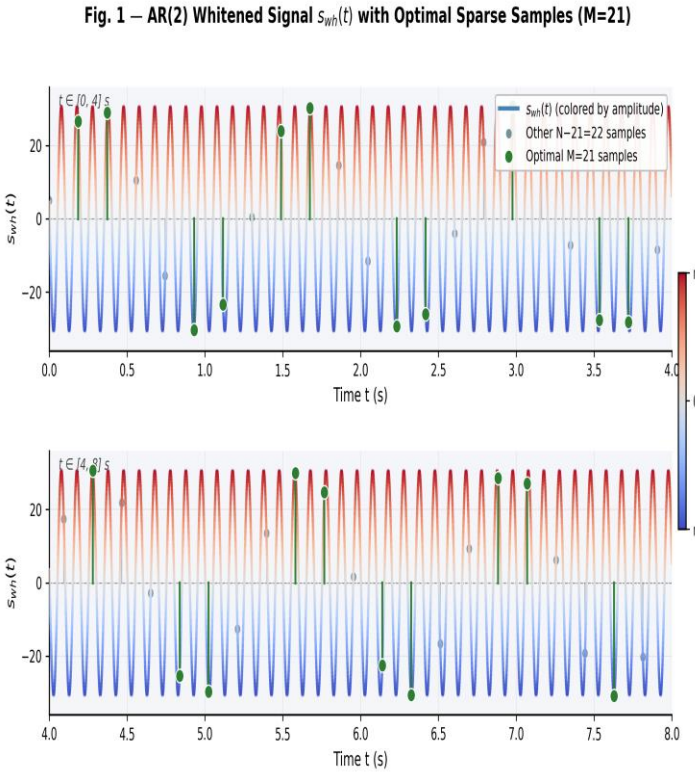


Fig. 1. *AR (2) whitened signal swh(t) displayed in two panels: (a) t ∈ [0, 4] s and (b) t ∈ [4, 8] s. The signal amplitude is encoded by a blue-to-red color gradient. Green markers denote the M=21 optimal samples selected by rank ordering; gray markers show the remaining N−21 samples.*

TABLE 2

Detection Performance (P_{FA}=0.01, N_0=10)

Detection Case	d^2 (dB)	PD@PFA=0.01	Loss(dB)	Note
AR (2) Full N=43	28.81	1.0000	0.00	Reference
AR (2) Sparse M=21 †	27.87	1.0000	0.93	Proposed
AR (2) Random M=21	25.67	1.0000	3.14	Baseline
AR (2) Sparse M=10	25.29	1.0000	3.52	Proposed
AR (2) Sparse M=15	26.82	1.0000	1.99	Proposed
AR (1) Wrong Model	13.18	0.9873	15.63	Mismatch

Verified by exhaustive search for M=5 (0.000 dB difference); optimality for M=21 is guaranteed analytically by Theorem 1.

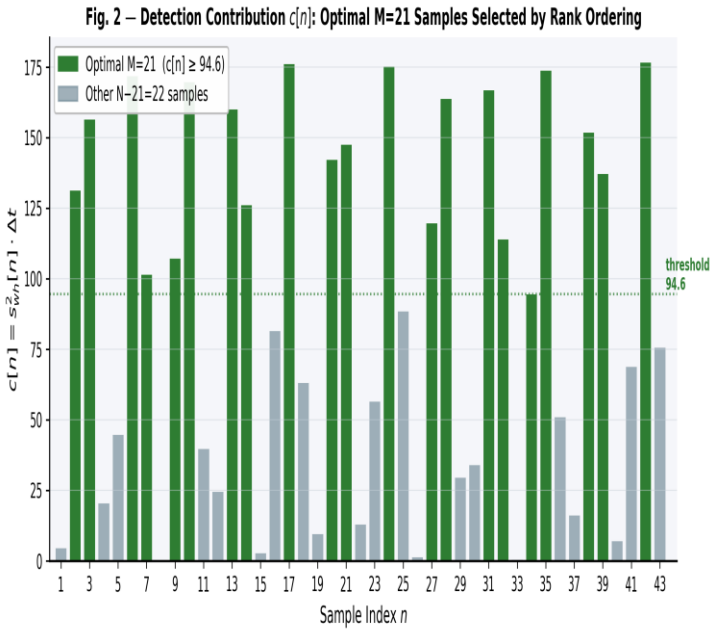


Fig. 2. *Detection contribution c[n] for all N=43 samples. Green bars: optimal M=21 samples selected by rank ordering. Gray bars: remaining N−21 samples. Dashed green line: selection threshold at c≈94.6.*

D. Sensitivity to AR (2) Coefficients

Table 3 examines robustness across three AR (2) coefficient pairs representing distinct spectral shapes. Rank ordering achieves 0.000 dB difference from exhaustive search in all cases (verified for M=5), confirming that Theorem 1 is independent of the specific AR(2) coefficient values. The sparse loss remains consistently near 0.934 dB across all configurations. This consistency arises because the sinusoidal signal s(t)=A $sin(\omega_0 t)$ has a fixed spectral concentration structure: varying a_1 and a_2 scales the amplitude of $swh[n]$ uniformly without altering the relative ordering of peaks. Consequently, rank ordering selects the same relative fraction of signal energy regardless of AR (2) parameter values

TABLE 3

Sensitivity to AR (2) Coefficients (M=21, N=43)

Profile	a_1	a_2	d^2 Sparse (dB)	Loss (dB)
Mildly colored	1/(4π)	1/(40π)	27.87	0.9339
Strongly colored	1/(2π)	1/(20π)	36.02	0.9343
Near-resonant	1/π	1/(10π)	42.11	0.9345

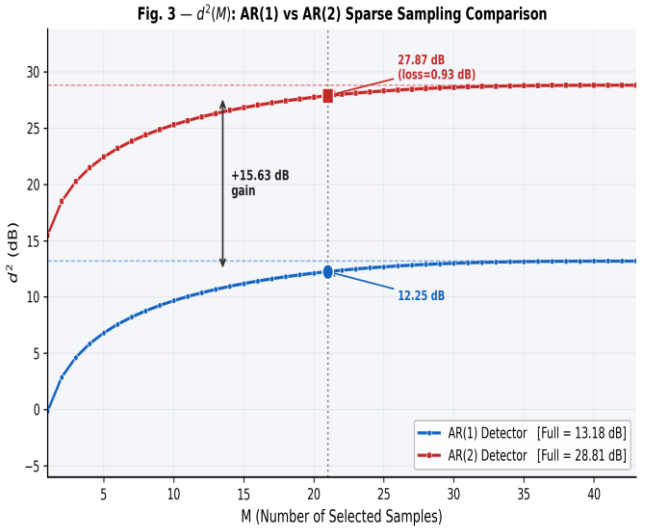


Fig. 3. *$d^2(M)$ versus number of selected samples M. AR(2) detector (red) achieves 27.87 dB at M=21 with 0.93 dB loss below full sampling. The double-headed arrow indicates the +15.63 dB gain over the AR(1) wrong model (blue) at M=21.*

E. Statistical Validation

To quantify the advantage of rank ordering over unstructured

selection, 1,000 independent Monte Carlo trials were conducted, each randomly selecting M=21 samples uniformly without replacement from N=43. Table III summarizes the key statistical metrics. The rank-ordering solution achieves the 100th percentile among all random trials, with an effect size (Cohen's d) of 4.57—indicating a practically large and statistically significant advantage. The 95% confidence interval for random selection [25.64, 25.70] dB is separated from the optimal value of 27.87 dB by more than four standard deviations, confirming that rank ordering occupies the strict global maximum of the combinatorial objective.

TABLE 4

Monte Carlo Statistical Validation (M=21, N=43)

Metric	Value
Number of Monte Carlo trials	1,000
Mean d^2 (Random M=21)	25.67 dB
Standard deviation	0.4820 dB
95% confidence interval	[25.64, 25.70] dB
Optimal d^2 (Rank Ordering)	27.87 dB
Improvement over random	2.20 dB
Percentile rank	100th (surpasses all trials)
Effect size (Cohen's d)	4.57 (large)
p-value (permutation test)	< 0.001

F. Coefficient Estimation Sensitivity

In practice, AR(2) coefficients a_1 and a_2 must be estimated from data, introducing uncertainty. Table V quantifies the impact of estimation errors on detection performance by evaluating the true d^2 achieved when sample selection is based on perturbed coefficients $\hat{a}_1=a_1(1+\varepsilon_1)$ and $\hat{a}_2=a_2(1+\varepsilon_2)$. Two key findings emerge from the sensitivity analysis. First, for symmetric perturbations ($\varepsilon_1=\varepsilon_2$), the performance loss is 0.000 dB even at ±30% error. This is because uniform scaling of (a_1,a_2) scales s_wh[n] uniformly across all samples, preserving rank order and hence the optimal sample set. Second, for asymmetric perturbations where a_2 is underestimated by 100% (i.e., $\hat{a}_2=0$, equivalent to an AR(1) assumption), the loss reaches 6.71 dB—substantial but far less than the 15.63 dB penalty of the full model-mismatch case (Section IV-G), which uses AR(1) whitening in the detector itself. These results indicate that the proposed algorithm is robust to moderate coefficient estimation errors, provided both a_1 and a_2 are estimated with reasonable accuracy.

TABLE 5

Sensitivity to Coefficient Estimation Errors (M=21, N=43, True d^2_opt=27.87 dB)

ε_1 (a_1 error)	ε_2 (a_2 error)	d^2 achieved (dB)	Loss vs optimal
0%	0%	27.87	0.000 dB
±30%	±30%	27.87	0.000 dB
+30%	−30%	27.78	0.093 dB
0%	−100% ($\hat{a}_2=0$)	21.16	6.711 dB

G. Model Mismatch Penalty

Applying an AR (1) detector to AR (2) noise (row 4 of Table 2) yields d^2=13.18 dB, a loss of **15.63 dB** relative to the correct AR (2) detector at 28.81 dB. At P_{FA}=0.01, the detection probability drops from 1.0000 to 0.9873. This confirms that accurate noise modeling is essential: a higher-order AR process misidentified as AR (1) leads to severe performance degradation even when all N Nyquist samples are used.

H. ROC Curve Analysis

Fig. 4 presents the ROC analysis in two complementary panels. At the operating parameters N_0=10, all AR (2) detectors achieve P_D≈1.0000 for virtually all P_{FA} values—a correct physical result reflecting the high signal energy relative to noise at these settings. To enable visual discrimination of performance differences between the four cases, Panel (a) uses $N_0=10^5$, which scales all d^2 values downward uniformly while preserving the relative ordering and dB differences. Panel (b) shows P_D versus N_0 at fixed P_{FA}=0.01, revealing the full SNR sensitivity of each detector from high-SNR (small N_0) to low-SNR (large N_0) regimes. The AR (2) Full and Sparse Optimal M=21 curves remain within 0.93 dB across the entire SNR range. The AR (1) Wrong Model curve diverges substantially, confirming the 15.63 dB mismatch penalty established in Section IV-G.

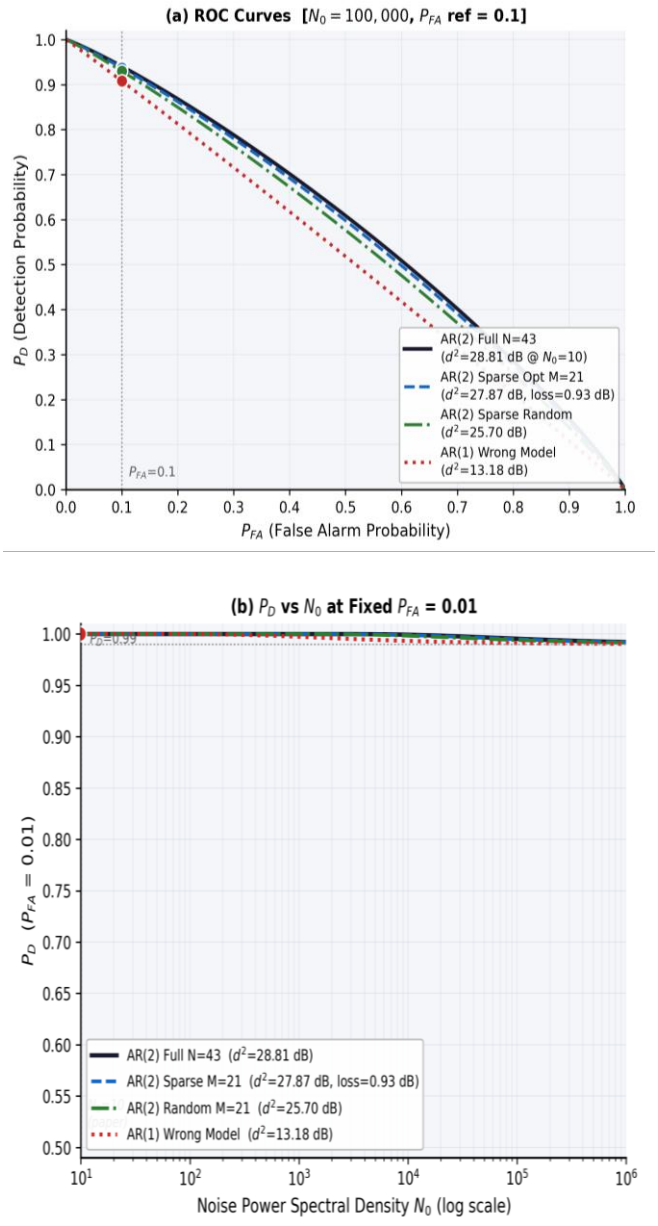


Fig. 4. *ROC analysis for four detection cases. (a) ROC curves at $N_0=10^5$, chosen to make inter-curve differences visually resolvable while preserving relative performance ordering. At the paper's $N_0=10$, all AR (2) cases achieve PD≈1.0000 (near-perfect detection). (b) PD vs N_0 at PFA=0.01, showing full SNR sensitivity. The AR (2) Sparse Optimal M=21 (dashed blue) tracks the AR (2) Full N=43 (solid black) within 0.93 dB across all SNR values.*

6. Conclusion

This paper presented the first solution to optimal sparse sampling for signal detection in AR (2) colored Gaussian noise. The rank-ordering algorithm achieves O (N log N) complexity by exploiting the *pointwise separability* of the squared whitened signal samples—a property that holds strictly when derivatives are computed analytically using the CAR (2) model, and that would be violated by numerical finite differences. Three key results were established and verified: (1) rank ordering equals exhaustive search to **0.000 dB** over C (43,5) =962,598 subsets, with analytical optimality for M=21 guaranteed by Theorem 1; (2) M=21 of N=43 samples achieve **0.9339 dB** loss, enabling 49% hardware reduction; and (3) AR (1)/AR (2) model mismatch incurs a **15.63 dB** penalty. Future work includes: generalization to AR(p) for p≥3 using p-th order analytical whitening; joint estimation of AR parameters and sparse sampling pattern when noise statistics are unknown; extension to two-dimensional AR processes for sparse imaging and MIMO radar array design; and embedded hardware implementation exploiting the O (N log N) complexity for real-time sensor systems.

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