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ON BI-TOPOLOGICAL SPACE

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ABSTRACT

Bi-topological spaces, a generalization of traditional topological spaces, provide a rich framework for studying the interplay between two distinct topologies defined on a single set. In a bi-topological space, open sets and neighborhoods are characterized by two separate systems of open sets, offering a nuanced understanding of continuity, convergence, and compactness properties. This abstract explores the foundational concepts of bi-topological spaces, including their definition, basic properties, and key theorems. Moreover, it discusses the significance of bi-topological spaces in various mathematical contexts, highlighting their applications in fields such as functional analysis, differential equations, and computer science. Through the lens of bi-topological spaces, these abstract illuminates the versatility and relevance of this mathematical structure in both theoretical investigations and practical applications.

الفضاءات التوبولوجية الثنائية

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الكلمات المفتاحية:
الفضاء التبولوجي الثنائي

الملخص

الفضاءات التوبولوجية الثنائية هي تعميم للفضاءات التوبولوجية التقليدية، وتوفر إطارًا غنيًا لدراسة التفاعل بين طوبولوجيتين مختلفتين معرفتين على مجموعة واحدة. في فضاء توبولوجي ثنائي، يتم تعريف المجموعات المفتوحة والتوبولوجي بواسطة نظامين منفصلين للمجموعات المفتوحة، مما يوفر فهمًا دقيقًا لخصائص الاستمرارية والتقارب والامتداد. يستكشف هذا الملخص المفاهيم الأساسية للفضاءات التوبولوجية الثنائية، بما في ذلك تعريفها وخصائصها الأساسية والنظريات الرئيسية. علاوة على ذلك، يناقش أهمية الفضاءات التوبولوجية الثنائية في مختلف السياقات الرياضية، مع تسليط الضوء على تطبيقاتها في مجالات مثل التحليل الوظيفي والمعادلات التفاضلية وعلوم الكمبيوتر. من خلال عدسة الفضاءات التوبولوجية الثنائية، يسلط هذا الملخص الضوء على تنوع وأهمية هذا البناء الرياضي في كل من البحوث النظرية والتطبيقات العملية.

1. Introduction

Topology is a branch of mathematics concerned with properties of spaces that are preserved under continuous deformations, such as stretching or bending without tearing. A fundamental concept in topology is the notion of a topological space, where a set is equipped with a topology that specifies which subsets are considered "open" and "closed." These open sets define the basic idea of nearness or proximity within the space.

Classical topology, however, often deals with a single notion of "open" and "closed" sets. Bi-topological spaces were initiated in 1963, by J. C. Kelly[1], offer a more general framework by equipping a set with two independent topologies, denoted by τ_1 and τ_2. This allows for exploring spaces where distinct concepts of closeness or openness coexist.

Lots of researchers were led to investigate other possible ways for defining properties in bi-soft topological spaces as [2], [3], [4], [5], [6], [7], [8], [9]

2. Bi-topological Spaces: Definitions and Basic Concepts

Definition 2.1 [7]

A bi-topological space is the triplet (X, τ_1, τ_2) where X is a non-empty set, τ_1 and τ_2 are two topologies on X.

Definition 2.2 [10]

A subset A of (X, τ_1, τ_2) is called

(i) $\boldsymbol{\tau_1\tau_2}$ **-open** if $A \in \tau_1 \cup \tau_2$,

the complement of $\tau_1\tau_2$ -open set is called **$\boldsymbol{\tau_1\tau_2}$ -closed set**

(ii) $\boldsymbol{\tau_1\tau_2}$ **-open** if $A = A_i \cup B_i$, where $A_i \in \tau_1$ and $B_i \in \tau_2$,

the complement of $\tau_{1,2}$ -open set is called $\boldsymbol{\tau_{1,2}}$ **-closed set.**

Example 3.1.

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A subset S of X is called $\tau_1\tau_2$ -open if S = H ∪ K such that H ∈ τ_1 and K ∈ τ_2 and the complement of $\tau_1\tau_2$-open is $\tau_1\tau_2$ -closed.

Definition 2.3 [5]

Let A be a subset of X. Then $\tau_1\tau_2$ Int (A) denotes the **$\tau_1\tau_2$ -interior** of A and is defined as the union of all $\tau_1\tau_2$ -open sets contained in A.

$\tau_1\tau_2$ -int(A)= ∪ {F: A⊆F and F is $\tau_1\tau_2$ -open set}

Example 2.1 [5]

Let X = {a, b, c}, τ_1 = {∅, X, {{a}} and τ_2 = {∅, X, {{b}}. The sets in {∅, X, {a}, {b}, {a, b}} are called $\tau_1\tau_2$- open and the sets in {∅, X, {b, c}, {a, c}, {c}} are called $\tau_1\tau_2$ closed.

Definition 2.4 [5]

Let (X, τ_1, τ_2) be a bi-topological space. and let $B \subseteq X$. A point $x \in X$ is called a **$\tau_1\tau_2$ -limit point** for B if $B \cap (U_x \backslash \{x\}) \neq \emptyset$ for any $\tau_1\tau_2$ -open set U_x containing x. The set of all $\tau_1\tau_2$ -limit points of B denoted by $(B')^{\tau_1\tau_2}$ and called the $\tau_1\tau_2$ -derived set of B.

Definition 2.5 [5]

Let A be a subset of X. Then $\tau_1\tau_2$-Cl(A) denotes the **<u>$\tau_1\tau_2$ -closure</u>** of A and is defined as the intersection of all $\tau_1\tau_2$-closed sets containing A.

$\tau_1\tau_2 \ - cl(A) = \cap \{F: A \subseteq F \ and \ F \ is \ \tau_1\tau_2 - closed \ set\}$ and $\tau_1\tau_2 \ (A) \subseteq \tau_1 - cl(A) \ and \ \tau_1\tau_2$ -$cl(A) \subseteq \tau_2$-$cl(A)$.

Example 2.2 [2]

Let $X = \{a, b, c, d\}, \tau_1 = \{X, \varphi, \{a\}, \{b\}, \{a, b\}\}, \tau_2 = \{X, \varphi, \{c\}, \{a, c\}\}$, and $A = \{a, d\}, then \ (A')^{\tau_1\tau_2} = \{d\}$.

Definition 2.6 [7]

Let (X, τ_1, τ_2) be a bi-topological space, any cover $F \subseteq \tau_1 \cup \tau_2$ of X is called **$\tau_1\tau_2$-open cover**. Further, if every $\tau_1\tau_2$-open cover of X possesses a finite subcover, then (X, τ_1, τ_2) is called **compact.**

Definition 2.7 [8]

Let (X, τ_1, τ_2) and (Y, ρ_1, ρ_2) be two bi-topological spaces, and let $f: (X, \tau_1, \tau_2) \rightarrow (Y, \rho_1, \rho_2)$be a map, then f is called **continuous** (open, closed) if the maps $f: (X, \tau_1) \rightarrow (Y, \rho_1)$ and $f: (X, \tau_2) \rightarrow (Y, \rho_2)$ are continuous (open, closed).

Definition 2.8 [1]

Let (X, τ_1, τ_2) and (Y, ρ_1, ρ_2) be two bi-topological spaces, and let $f: (X, \tau_1, \tau_2) \rightarrow (Y, \rho_1, \rho_2)$ be a map, then f is called **homeomorphism** if the maps $f: (X, \tau_1) \rightarrow (Y, \rho_1) \ and \ f: (X, \tau_2) \rightarrow (Y, \rho_2)$ are homeomorphisms

Definition 2.9 [11]

A bi-topological space (X, τ_1, τ_2) is said to be **connected** if and only if X cannot be expressed as the union of two non-empty disjoint sets A and B such that A is τ_1-open and B is. τ_2-open. When X can be so expressed, we write $X = A \ \backslash B$ and call this a **disconnection** of *X*.

Definition 2.10 [12]

Let (X, τ_1, τ_2) be a bi-topological space and let $(x_n)_{n=1}^{\infty} = \{x_1, x_2, \dots \dots, x_n, \dots .\}$ be a **sequence** in X. We say $(x_n)_{n=1}^{\infty}$ converge to a point $x \in X \ if \ (x_n)_{n=1}^{\infty}$ converge to x in (X, τ_1) and $(x_n)_{n=1}^{\infty}$ converge to x in (X, τ_2).

Example 2.3 [12]

Let $X = IN$, and let (X, τ_1, τ_2) be a bi-topological space were $\tau_1 = \{X, \emptyset\} and \ \tau_2 = \{X, \emptyset, \{2\}, \{2,3\}, \{2,3,4\}, \dots \dots\}$, $(n)_{n=1}^{\infty} = \{1,2,3, \dots \dots .\}, then \ (n)_{n=1}^{\infty} \rightarrow 1, but \ (n)_{n=1}^{\infty} \nrightarrow 2$.

3. Separation Axioms in Bi-topological Spaces

Definition 3.1 [12]

A bi-topological space (X, τ_1, τ_2) is called **T_0** space if $\forall \ x, y \in X$ with $x \neq y$ then $\exists U \in S \cup T$ such that $x \in U, y \notin U \ and \ x \notin U, y \in U$.

Definition 3.2 [12]

A bi-topological space (X, τ_1, τ_2) is called **T_1** space if $\forall \ x, y \in X$ with $x \neq y$ then $\exists U \in S$ and $U \in T$ such that $x \in U, y \notin U \ or \ x \notin U, y \in U$.

Definition 3.3 [12]

A bi-topological space (X, τ_1, τ_2) is called **T_2** space if $\forall \ x, y \in X$ with $x \neq y$ then $\exists U \in S, V \in T$ such that $x \in U, y \in V$ and $U \cap V = \emptyset$

4. Pairwise In Bi-topological Spaces

Definition 4.1 [2]

Let (X, τ_1, τ_2) and (Y, ρ_1, ρ_2) be two bi-topological spaces and let $f: (X, \tau_1, \tau_2) \rightarrow (Y, \rho_1, \rho_2)$ be a map, then f is called:

1) **Pairwise continuous** (p-continuous for short) if $f^{-1}(V) \in \tau_1 \cup \tau_2 \ for \ any \ V \in \rho_1 \cup \rho_2$.

2) **Pairwise open** (p-open for short) if $f(V) \in \rho_1 \cup \rho_2$ for any $V \in \tau_1 \cup \tau_2$.

3) **Pairwise closed** (p-closed for short) if *f*(F) is $\rho_1\rho_2$-closed set in (Y, ρ_1, ρ_2) for any $\tau_1\tau_2$-closed set F in (X, τ_1, τ_2).

4) **Pairwise homeomorphism** (p-homeomorphism for short) if f is a bijective function and *f*, f^{-1} are p-continuous.

Theorem 4.1. [12]

Let $f: (X, \tau_1, \tau_2) \rightarrow (Y, \rho_1, \rho_2)$ be a continuous (open, closed, homeomorphism), then f is p-continuous (p-open, p-closed, p-homeomorphism).

5. Pairwise Separation Axioms in Bi-topological Spaces

Definition 5.1. [7]

A bi-topological space (X, τ_1, τ_2) is said to be **pair-wise T_0** if, for each pair of distinct points of X, there is a τ_1 -open set or a τ_2-open set containing one of the points, but not the other.

Definition 5.2. [6]

A bi-topological space (X, τ_1, τ_2) is said to be **pair-wise T_1**, if for each pair of distinct points x, y there exist $U \in \tau_1, \tau_2 \in Q$ such that $x_1 \in U, x_2 \notin V \ and \ x \notin U, y \in V$.

Definition 5.3. [7]

The bi-topological space (X, τ_1, τ_2) is said to be **dually <u>Hausdorff</u>** <u>(T_2)</u>, if for any points $x_1 \in X, \ x_2 \in X, \ x_1 \neq x_2$ there exist $G_1 \in \tau_1, \ G_2 \in \tau_2$ such that $x_1 \in \tau_1, x_2 \in \tau_2$, $G_1 \cap G_2 = \emptyset$.

6. Conclusion

In conclusion, the paper aims to provide a powerful framework for studying topological structures that arise in various mathematical contexts. By considering two distinct topologies on a single set, bi-topological spaces offer a nuanced understanding of the interplay between different types of open sets and neighborhoods. This allows for a more refined analysis of continuity, convergence, and compactness properties than what is possible in traditional topological spaces. Furthermore, the study of bi-topological spaces has applications in diverse areas such as functional analysis, differential equations, and computer science, making them a valuable tool for both theoretical investigations and practical applications. As research in this area continues to advance, the rich structure and versatility of bi-topological spaces are likely to yield further insights and discoveries in the broader field of mathematics.

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