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# Further Remarks On Somewhere Dense Sets

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**ABSTRACT**

In this article, we prove that a topological space X is strongly hyperconnected if and only if any somewhere dense set in X is open, in addition we investigate some conditions that make sets somewhere dense in subspaces, finally, we show that any topological space defined on infinite set X has SD-cover with no proper subcover.

# ملاحظات اضافية على المجموعات الكثيفة في مكان ما

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**الكلمات المفتاحية:**
الفضاء التبولوجى والتعميمات
المجموعات الكثيفة في مكان ما
الفضاءات الجزئية
الغطاء
الاعداد الاصلية
علاقات التكافؤ

**الملخص**

في هذه المقالة، نثبت ان الفضاء التبولوجى X يكون strongly hyperconnected إذا وفقط إذا كانت أي مجموعة كثيفة في مكان ما تكون مفتوحة، بالإضافة الى ذلك ندرس بعض الشروط التي تجعل المجموعات كثيفة في مكان ما في الفضاءات الجزئية، وأخيرا نبين بان أي فضاء تبولوجى معرف على مجموعة غير منتهية له غطاء من النوع SD وليس له غطاء جزئي فعلى.

**Introduction:**

Using the closure and the interior operators in topological space, different types of generalized open sets have been defined as; α-set, semi-open set, preopen set, β-open set, b-open set and somewhere dense [1,2,3,4,5,6]. The concept of somewhere dense set was due to Pugh [6], where a set A is somewhere dense if the interior of its closure is non-empty, clearly somewhere dense set is a generalization of both open set and dense set. In 2017, Alshami [7] provided the properties of somewhere dense sets, and he introduced the axiom of $ST_1$, then with Noiri they defined the notion of SD-cover and use it to introduced compactness and lindelöfness via somewhere dense sets, see [8,9].

A space is hyperconnected [10], if every two non-empty open sets intersect; equivalently if any non-empty open set is dense, while a space is submaximal [11] if any dense set is open, and in 1994 Rose and Mahmoud [12] showed that a space is submaximal if and only if every preopen set is open, for more details see [13,14,15,16,17,18,19,20,21]. Recently, Alshami [7,8] defined strongly hyperconnected space as a hyperconnected submaximal space, and he charactrized this space using the notion of somewhere dense sets [8].

The main goal of this article, is to continue studying further properties on somewhere dense sets, and imporve some of the results given by Alshami and Noiri [7,8] regarding strongly hyperconnected space. Here we give solutions to the following questions, equiped with some examples:

**Question 1.** Find the necessary and sufficient condition under which every somewhere dense set is open?.

**Question 2.** If (X,τ) is a topological space and Y⊆ X: find conditions under which set in the subspace Y is somewhere dense in X?.

**Question 3.** If (X,τ) is a topological space where X is infinite set: find a cover for X by somewhere dense sets (SD-cover) which has no proper subcover.

The article is divided into four sections: somewhere dense sets in strongly hyperconnected space, somewhere dense sets in subspaces, SD-covers and conclusion. Throughout this article X or (X,τ)

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represents topological space and for a subset A of a space X, $\overline{A}$ and $A^o$ denote the closure and the interior of A; respectively. Moreoever, X/A (or $A^c$), A/B and P(X) denote the complement of the set A in X, the difference of A and B, and the power set of X; respectively, while $\sim$ denotes the equivalence relation, and $\chi_0$, $\chi_1$ are the cardinality of the natural numbers $\mathbb{N}$ and the real numbers $\mathbb{R}$; respectively.

## 2. Somewhere Dense Sets In Strongly Hyperconnected Spaces

This section, consists basic definitions, theorems and some properties regarding somewhere dense sets needed in this work, and then we give a complete answer for question 1, by studying the statament when any somewhere dense set is open.

**Definition 2.1.** [7] A subset B of a topological space (X,τ) is called somewhere dense (briefly s-dense) if the interior of its closure is non-empty, i.e. $\overline{B}^o \neq \phi$. The complement of s-dense set is called closed somewhere dense (briefly cs-dense), and the collection of all s-dense sets in X is denoted by S(τ).

**Corollary 2.1.** [7] In a topological space (X,τ), we have :

*i.* any open set is s-dense.
*ii.* any dense set is s-dense
*iii.* any set in X that contains a s-dense set is s-dense.

**Theorem 2.1.** [7] Every subset of a space (X,τ) is s-dense or cs-dense.

**Theorem 2.2.** For a non-discrete topological space (X,τ), the following are equivalent:

1) $S(\tau)=\tau/\{\phi\}$.
2) $D(\tau)=\tau/\{\phi\}$, where D(τ) denotes the collection of all dense sets in (X,τ).

**PROOF.**

1) ⟹ 2) Let (X,τ) be a non-discrete space that satisfies $S(\tau)=\tau/\{\phi\}$, and suppose that $D(\tau)\neq\tau/\{\phi\}$. From corollary 2.1.(*ii*) any dense set is s-dense, then from assumption any dense set is open so $D(\tau)\subseteq\tau$. So suppose that A is a non-empty open set but not dense in X, hence $\overline{A}\neq X$, therefor $\overline{A}^c \neq \phi$. Now if x∈X, then $x\in\overline{A}$ or $x\in\overline{A}^c$. In the case when $x\in\overline{A}$, we have A is open so it is s-dense, and since $A\subseteq\overline{A}$ from corollary 2.1.(*iii*) we obtain $\overline{A}$ is also s-dense, so it is open. Since $\overline{A}^c$ is open, so it is s-dense, then $\overline{A}^c\cup\{x\}$ is also s-dense, so it is open , and the intersection of two open sets is open, hence $\{x\}=\overline{A}\cap\left(\overline{A}^c\cup\{x\}\right)$ is open. In the second case when $x\in\overline{A}^c$, and by similar reasons the sets $\overline{A}^c$ and $\overline{A}\cup\{x\}$ are both open, then thier intersection is also open, hence $\{x\}=\overline{A}^c\cap\left(\overline{A}\cup\{x\}\right)$ is open. Thus {x} is open for any x∈X, hence τ is the discrete topology on X, which contradicts the assumption. Thus complete the prove, and $D(\tau)=\tau/\{\phi\}$.

2) ⟹ 1) Let (X,τ) be a non-discrete space that satisfies $D(\tau)=\tau/\{\phi\}$, and suppose A is a s-dense in X, then $\overline{A}^o$ is a non-empty open set in X, therefore $\overline{A}^o$ is dense in X, but $\overline{A}^c$ is open and $\overline{A}^o\cap\overline{A}^c=\phi$, since $\overline{A}^o$ is dense we have $\overline{A}^c=\phi$, so $\overline{A}=X$, thus A is dense, so it is open. Since, any open set is s-dense, we obtain $S(\tau)=\tau/\{\phi\}$.

**Definition 2.2.** [10,11] A topological space (X,τ) is called:

1) Submaximal if any dense set in X is open.
2) hyperconnected if any non-empty open set in X is dense.

**Corollary 2.2.** If (X,τ) is submaximal and hyperconnected, then $D(\tau)=\tau/\{\phi\}$.

**Definition 2.3.** [7] A topological space (X,τ) is called strongly hyperconnected if non-empty open sets are coincide with dense sets, equivalently; if $D(\tau)=\tau/\{\phi\}$.

**Corollary 2.3.** For a space (X,τ) the following are equivalent:

1) X is strongly hyperconnected space.
2) X is submaximal and hyperconnected space.
3) Any s-dense set in X is open.
4) $D(\tau)=\tau/\{\phi\}$.
5) $S(\tau)=\tau/\{\phi\}$.

**Examples 2.1.**

*i.* If (X,τ) is a trivial topological space where X has more than one element, then X is not strongly hyperconnected since $S(\tau)=P(X)/\{\phi\}=D(\tau)$.
*ii.* If $(\mathbb{R},\tau)$ is a topological space where $\tau=\{U\subseteq\mathbb{R}: 0\in U\}\cup\{\phi\}$, then X is a strongly hyperconnected, since $S(\tau)=\tau/\{\phi\}=D(\tau)$.
*iii.* The topological space $(\mathbb{R},\tau)$ where $\tau=\{U\subseteq\mathbb{R}: 0\notin U\}\cup\{\mathbb{R}\}$ is submaximal but not strongly hyperconnected space, since $S(\tau)=P(\mathbb{R})/\{\{0\},\phi\}$, but $D(\tau)=\{\mathbb{R}, \mathbb{R}/\{0\}\}$.

## 3. Somewhere Dense Sets In Subspaces

Here we answer question 2 by investigating some conditions in topological space X that make a set in a subspace somewhere dense in X.

**Corollary 3.1.** [11] Let (X,τ) be a topological space, Y be a subspace of X and A⊆Y, then:

1) $\overline{A}^Y\subseteq\overline{A}$ and $\overline{A}^Y=\overline{A}\cap Y$ (where $\overline{A}^Y$ is the closure of A with respect to the subspace Y).
2) $A^o\subseteq A^{o^Y}$ and $A^o=A^{o^Y}\cap Y^o$ (where $A^{o^Y}$ is the interior of A with respect to the subspace Y).

**Lemma 3.1.** Let (X,τ) be a topological space, and Y be a subspace of X, then:

1) If A⊆Y and Y is closed, then $\overline{A}^Y=\overline{A}$.
2) If A⊆Y and Y is open, then $A^{o^Y}=A^o$.
3) If A⊆Y and Y is clopen, then $\overline{A}^Y=\overline{A}$ and $A^{o^Y}=A^o$ .

**PROOF.**

1) Since A⊆Y we have $\overline{A}^Y=\overline{A}\cap Y\subseteq\overline{A}$. Y and $\overline{A}$ are closed sets, then $\overline{A}^Y$ is closed set containing A, so $\overline{A}\subseteq\overline{A}^Y$, we obtain $\overline{A}^Y=\overline{A}$.
2) Since A⊆Y we have $A^o\subseteq A^{o^Y}\subseteq A$ . Y is open, then $\tau_Y\subseteq\tau$, so $A^{o^Y}$ is open set contained in A, we obtain $A^{o^Y}=A^o$ .
3) Direct from (1) and (2).

**Theorem 3.1.** Let (X,τ) be a topological space, Y be a subspace of X, A⊆Y then:

1) If Y is closed and A is s-dense in X, then A is s-dense in Y.
2) If Y is open and A is s-dense in Y, then A is s-dense in X.
3) If Y is clopen, then A is s-dense in Y iff A is s-dense in X.

**PROOF.**

1) Since A is s-dense in X, we have $\overline{A}^o\neq\phi$. Y is closed then $\overline{A}\subseteq Y$, so $\overline{A}^o\subseteq(\overline{A})^{o^Y}$ and from the prevoius lemma (1) we obtain $\overline{A}=\overline{A}^Y$ , then $\overline{A}^o\subseteq((\overline{A}^Y)^{o^Y})$ . Since $\overline{A}^o\neq\phi$ we obtain $((\overline{A}^Y)^{o^Y})\neq\phi$, thus A is s-dense in Y.
2) Since A is s-dense in Y, then from the previous lemma (2) we have $(\overline{A}^Y)^o=(\overline{A}^Y)^{o^Y}\neq\phi$. Since $\overline{A}^Y\subseteq\overline{A}$ we have $(\overline{A}^Y)^o\subseteq\overline{A}^o$, then $(\overline{A}^Y)^{o^Y}=(\overline{A}^Y)^o\subseteq\overline{A}^o$, but $(\overline{A}^Y)^{o^Y}\neq\phi$, thus $\overline{A}^o\neq\phi$, i.e. A is s-dense in X.
3) Direct from (1) and (2).

**Example 3.1.** In the space $(\mathbb{R},\tau)$ where $\tau=\{U\subseteq\mathbb{R}: 0\in U\}\cup\{\phi\}$, we have $S(\mathbb{R})=\tau/\{\phi\}$, so if $Y=\mathbb{R}/\{0\}$ and A={1}, then Y is closed and A is s-dense in Y, while A is not s-dense in X since the relative topology on Y is the discrete topology.

**Corollary 3.2.** Let (X,τ) be a topological space, and A be a subset of X with non-empty interior, then A is s-dense.

**PROOF.** Since $A\subseteq\overline{A}$ we have $A^o\subseteq\overline{A}^o$, and $A^o$=A implies that $\overline{A}^o\neq\phi$.

**Corollary 3.3.** Let (X,τ) be a topological space, and A be a subset of X with non-empty interior, then A is s-dense in any subspace Y containing A.

**PROOF.** Suppose Y is a subspace of X, and A⊆Y then $A\subseteq\overline{A}^Y\subseteq Y$ , so we obtain $A^o\subseteq(\overline{A}^Y)^o\subseteq(\overline{A}^Y)^{o^Y}$ , therefore $((\overline{A}^Y)^{o^Y})\neq\phi$, thus A is s-dense in Y.

**Definition 3.1.** [22] A subset B of a space (X, τ) is called regular closed (briefly r-closed) if $B=\overline{B^o}$. Note that, any r-closed set is closed.

**Theorem 3.2.** Let (X,τ) be a topological space, Y be an r-closed subspace of X, A⊆Y, then A is s-dense in Y iff A is s-dense in X.

**PROOF.**

$\Leftarrow$ Direct from theorem (3.1), since any r-closed set is closed.

$\Rightarrow$ Suppose A is s-dense in Y, then $((\overline{A}^{Y})^{o^{Y}}) \neq \phi$, and since Y is r-closed so closed we obtain $\overline{A} = \overline{A}^{Y} \subseteq Y$, so $(\overline{A})^{o^{Y}} = ((\overline{A}^{Y})^{o^{Y}}) \neq \phi$. From corollary (3.1.(2)) we have $\overline{A}^{o} = (\overline{A})^{o^{Y}} \cap Y^{o}$. Suppose $\overline{A}^{o} = \phi$, then $(\overline{A})^{o^{Y}} \cap Y^{o} = \phi$, and since $(\overline{A})^{o^{Y}} \neq \phi$, there exists an open set W such that $W \cap Y^{c} \neq \phi$, $W \cap \overline{A} \neq \phi$ and $W \cap Y \subseteq \overline{A}$, therefore $W \cap Y \subseteq (\overline{A})^{o^{Y}}$. Now Y is r-closed set, so we have $Y^{o} \neq \phi$, and since $(\overline{A})^{o^{Y}} \cap Y^{o} = \phi$ we obtain $W \cap Y^{o} = \phi$ (because $W \subseteq (\overline{A})^{o^{Y}}$), so $Y^{o} \subseteq W^{c}$, hence $\overline{Y^{o}} \subseteq \overline{W^{c}} = W^{c}$, since Y is r-closed we get $\overline{Y^{o}} = Y \subseteq W^{c}$, so $Y \cap W = \phi$, which is a contradiction since $W \cap \overline{A} \neq \phi$ and $\overline{A} \subseteq Y$. Hence $\overline{A}^{o} \neq \phi$, thus A is s-dense in X.

## 4. SD-Covers

In the present section, we answer question 3 by proving that any topological space defined on infinite set X has a cover by somewhere dense sets with no proper subcover.

**Definition 4.1.** [9] If (X,τ) is a topological space, then a cover for X by s-dense subsets is called SD-cover for X.

**Remark 4.1.** Any cover for a space X is SD-cover, but the converse is not true.

**Examples 4.1.**

*i.* If (X,τ) is a trivial topological space, then $S(\tau)=P(X)/\{\phi\}$, so when X is uncountable then the collection of all singeltons is SD-cover for X with no countable subcover.

*ii.* If (X,τ) is the cofinite topological space, then $S(\tau)=\{U\subseteq X: U \text{ is infinite}\}$, so if $X = A\cup B$ where A is countable, B is uncountable and $A\cap B = \phi$, then $\{A\cup\{x\}\}_{x\in B}$ is an SD-cover for X with no countable subcover.

**Lemma 4.1.** For any infinite set Z there exists a subset A of Z such that $|Z|=|A|=|A^{c}|$.

**PROOF.** By the mathematical indiction, we have:

1) In the case when $|Z|=\chi_0$, then $Z \sim \mathbb{N} \sim E \sim E^{c}$; where $\mathbb{N}$, and E are the natural numbers and the even numbers, then there exists a bijection function $F:\mathbb{N}\to Z$. Now set A=F(E), then $Z \sim \mathbb{N} \sim E \sim F(E)=A$, so $|Z|=|A|=\chi_0$, now since $\mathbb{N} \sim E^{c} \sim F(E^{c}) = A^{c}$, then $|A^{c}|=\chi_0$.
2) In the case when $|Z|=\chi_1$, then $Z \sim \mathbb{R} \sim \mathbb{R}^{+} \sim (\mathbb{R}^{+})^{c}$; where $\mathbb{R}$ and $\mathbb{R}^{+}$ are the real numbers and the positive real numbers, then there exists a bijection function $F: \mathbb{R} \to Z$. Now set $A=F(\mathbb{R}^{+})$, then $Z \sim \mathbb{R} \sim \mathbb{R}^{+} \sim F(\mathbb{R}^{+})=A$, so $|Z|=|A|=\chi_1$, now since $\mathbb{R} \sim (\mathbb{R}^{+})^{c} \sim F((\mathbb{R}^{+})^{c}) = A^{c}$, then $|A^{c}|=\chi_1$.
3) Suppose $|Z|=\chi_n$ and A is a subset of Z such that $|Z|=|A|=|A^{c}|=\chi_n$. Let Y be a set with cardinal number $\chi_{n+1}$, then $P(Z) \sim Y$, and $Y\sim P(Z) \sim P(A) \sim P(A^{c})$. Let F be a bijection function $F: P(Z) \to Y$, and set B=F(P(A)). Then $B \sim P(A) \sim Y$. Now since $P(A^{c})/\{\phi\}\subseteq (P(A))^{c}$ so $F(P(A^{c})/\{\phi\})\subseteq B^{c}$ and $F(P(A^{c})/\{\phi\})\sim F(P(A^{c}))\sim P(A^{c})\sim P(Z) \sim Y$. We get $F(P(A^{c})) \subseteq B^{c} \subseteq Y$, then $B^{c}\sim Y$. Thus $|Y|=|B|=|B^{c}|=\chi_{n+1}$.

This complete the prove.

**Theorem 4.1.** If (X,τ) is a topological space where X is infinite, then (X,τ) has a SD-cover with no proper subcover.

**PROOF.** From lemma (4.1) there exists a subset A of X such that $|X|=|A|=|A^{c}|$. So A and $A^{c}$are infinite, and form theorem (2.1) at least A or $A^{c}$ is s-dense. Suppose A is s-dense, then from corollary (2.1.(*iii*)) $A\cup\{x\}$ is also s-dense for any $x \in A^{c}$, hence $\{A\cup\{x\}\}_{x\in A^{c}}$ is SD-cover for X with no proper subcover. Similarly in the case when $A^{c}$ is s-dense, we have $\{A^{c}\cup\{x\}\}_{x\in A}$ is SD-cover for X with no proper subcover.

## 5. Conclusion

In this article, we investigate some further topological properties on somewhere dense sets, and we have obtained few results; as follows: If (X,τ) is a topological space; then any somewhere dense set in X is open if and only if X is strongly hyperconnected, and if X is infinite set, then there exists a cover for X by somewhere dense sets with no proper subcover. Moreover, if Y is a subspace and $A\subseteq Y\subseteq X$; then: In the case when the subspace Y is open (closed) in X, if A is somewhere dense in Y (X), then A is somewhere dense in X (Y), while in the case when the subspace Y is regular closed in X, A is somewhere dense in Y if and only if A is somewhere dense in X.

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